THE CAPPELL-SHANESON EXAMPLE

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Introduction

In this note we shall be considering the quaternion group $Q_8 = \{x,y \mid x^2 = y^2 = (xy)^2\}$, denoting it by π. From [6] we have that $\tilde{K}_0(\mathbb{Z}(\pi)) = \mathbb{Z}/2$ with generator the Euler characteristic of the trivial $\mathbb{Z}(\pi)$-module $\mathbb{Z}/3$, which we denote $<3^{++}>$.

From e.g. [5] we have that $L_1^h(\mathbb{Z}(\pi)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ with a canonical non-zero class [A] given by the image of the non-trivial class in $\hat{H}_{ev}(\mathbb{Z}/2;\tilde{K}_0(\mathbb{Z}(\pi)))$ in the Ranicki-Rothenberg exact sequence.

Now, π acts freely on S^3, and in [2] Cappell and Shaneson prove that the surgery obstruction in $L_1^h(\mathbb{Z}(\pi))$ of the map

I.1 $$1 \times f \;:\; S^3/\pi \times K^{4j+2} \longrightarrow S^3/\pi \times S^{4j+2}$$

is non-trivial, and given by [A], with f representing the simply-connected Kervaire problem. However, their proof proceeds by an intricate "peeling" argument, and it has seemed desirable for a number of reasons to have a purely algebraic proof of their result.

In the current volume Jim Davis' paper [3] is concerned with this question, and provides a general recognition principle by which one can decide if a symmetric Poincaré structure on a chain complex (necessary for the application of the surgery product formula of [7]) is "geometric". Also Hambleton and Ranicki in as yet unpublished joint work have obtained other algebraic proofs based on "peeling".

In this note we first construct a 3-dimensional chain complex C_* of f.g. free $Z(\pi)$-modules and a chain equivalence $\phi : C^{3-*} \longrightarrow C_*$. We then analyze the class of ϕ in $\mathbb{Z} \otimes_{\mathbb{Z}(\pi)} (C_* \otimes C_*)$ and note that this class determines ϕ up to chain homotopy. Comparing this class with that of the diagonal chain map gives that ϕ is the base map of a suitable symmetric Poincaré structure on C_*, so taking the product of (C_*,ϕ) and the algebraic Kervaire problem gives an explicit quadratic Poincaré complex whose surgery obstruction is that of I.1.

This problem is then quickly evaluated (the procedures used here may have independent interest) and the Cappell-Shaneson example is the result. Indeed, by way of illustrating this last comment the final section indicates how to extend these results to the remaining compact space forms.

A. The complex C_* and the map ϕ

There is exactly one chain homotopy class of finitely generated free $\mathbb{Z}(\pi)$-module chain complexes with the homology of S^3 (since the set of such classes is given by $U(\mathbb{Z}/8)/\langle\pm 1, \mathrm{im} T\rangle = \{1\}$, with T the Swan subgroup, see e.g. [4]). A representative is given in [1] and is specified as follows

A.1

i	C_i	Generators	∂
0	$\mathbb{Z}(\pi)$	a	0
1	$\mathbb{Z}(\pi)\oplus\mathbb{Z}(\pi)$	b	$(x-1)a$
		b'	$(y-1)a$
2	$\mathbb{Z}(\pi)\oplus\mathbb{Z}(\pi)$	c	$(1+x)b-(y+1)b'$
		c'	$(xy+1)b+(y+1)b'$
3	$\mathbb{Z}(\pi)$	e	$(x-1)c-(xy-1)c'$.

Then C* is specified by the formulae

A.2
$$\delta(c^*) = (x^3-1)e^*$$
$$\delta(c'^*) = -(yx-1)e^*$$
$$\delta(b^*) = (1+x^3)c^* + (yx+1)c'^*$$
$$\delta(b'^*) = -(y^3+1)c^* + (x^3-1)c'^*$$
$$\delta(a^*) = (x^3-1)b^* + (y^3-1)b'^* \quad .$$

The chain equivalence $\phi: C^{3-*} \longrightarrow C_*$ is given by the equations

A.3
$$\phi(e^*) = a$$
$$\phi(c^*) = -x^3 b$$
$$\phi(c'^*) = -(yb+b')$$
$$\phi(b^*) = (yx-(y^3+1)(1+x))c' + (2y^3-y)c$$
$$\phi(b'^*) = x^{-1}c'$$
$$\phi(a^*) = (2y^3-y)e \quad .$$

B. ϕ is the beginning map in the Mishchenko-Ranicki symmetric structure on S^3/π

The set of chain homotopy classes of $\mathbb{Z}(\pi)$-module chain maps

$$\phi : C^{3-*} \longrightarrow C_*$$

is in 1-1 correspondence with $H_3(\mathbb{Z}\otimes_{\mathbb{Z}(\pi)}(C_*\otimes C_*))$, where $\mathbb{Z}(\pi)$ acts on $C_*\otimes C_*$ via the diagonal map

B.1 $$\Delta : \mathbb{Z}(\pi) \longrightarrow \mathbb{Z}(\pi)\otimes\mathbb{Z}(\pi) = \mathbb{Z}(\pi\times\pi) \; ; \; g \longmapsto g\otimes g \quad .$$

This is well known, see e.g. [7].

Proposition B.2 $H_3(\mathbb{Z}\otimes_{\mathbb{Z}(\pi)}(C_*\otimes C_*)) = \mathbb{Z}\oplus\mathbb{Z}$ with generators A,B (say) and the projection $p:C_*\otimes C_* \longrightarrow \mathbb{Z}\otimes_{\mathbb{Z}(\pi)}(C_*\otimes C_*)$ induces an injection

$$p_* : H_3(C_*\otimes C_*) \longrightarrow H_3(\mathbb{Z}\otimes_{\mathbb{Z}(\pi)}(C_*\otimes C_*)) \; ;$$

$$e\otimes 1 + 1\otimes e \longmapsto 8A \; , \; e\otimes 1 \longmapsto B \; .$$

Proof: There is a spectral sequence converging to $H_*(\mathbb{Z}\otimes_{\mathbb{Z}(\pi)}(C_*\otimes C_*))$ with

$$E^2_{i,j} = H_i(\pi, H_j(C_*\otimes C_*))$$

so $E^2_{i,j} \neq 0$ only for $j = 0,3,6$. Moreover $H_4(\pi,\mathbb{Z}) = 0$, so $d_4 = 0 : E_{4,0} \longrightarrow E_{0,3}$ and $H_3(\mathbb{Z}\otimes_{\mathbb{Z}(\pi)}(C_*\otimes C_*))$ is given as an extension

B.3 $$0 \longrightarrow E_{0,3} \xrightarrow{\;i\;} H_3(\mathbb{Z}\otimes_{\mathbb{Z}(\pi)}(C_*\otimes C_*)) \longrightarrow E_{3,0} \longrightarrow 0$$

where $E_{0,3} = \mathbb{Z}\oplus\mathbb{Z}$ and $E_{3,0} = \mathbb{Z}/8$. Moreover i in B.3 is the map p_*.

To determine the extension in B.3 note that the geometric diagonal $d:S^3 \longrightarrow S^3\times S^3$ is π-equivariant, so that there is an algebraic chain approximation such that the diagram below commutes

B.4 $$\begin{array}{ccc} C_* & \xrightarrow{\;d_\#\;} & C_*\otimes C_* \\ \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle p} \\ \mathbb{Z}\otimes_{\mathbb{Z}(\pi)} C_* & \xrightarrow{\;\hat{d}_\#\;} & \mathbb{Z}\otimes_{\mathbb{Z}(\pi)}(C_*\otimes C_*) \end{array} \quad .$$

But $H_3(C_*) = \mathbb{Z}$ and $H_3(\mathbb{Z}\otimes_{\mathbb{Z}(\pi)}C_*) = \mathbb{Z}$, with p_{0*} multiplication by 8. On the other hand $d_*:H_3(C_*) \longrightarrow H_3(C_*\otimes C_*)$ sends the generator e to $e\otimes 1 + 1\otimes e$, and B.2 follows.

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Corollary B.5 The map ϕ defined in A.3 is chain homotopic to the map corresponding to the diagonal in B.4. More exactly $[\phi] = \hat{d}_*(f)$ where f is a generator of $H_3(\mathbb{Z}\otimes_{\mathbb{Z}(\pi)}C_*)$.

Proof: It suffices to show that $[\phi]\in H_3(C_*\otimes C_*)$ is $d_*(e)$ from B.2. But this is the case if and only if

$$(\phi)_* : H^3(C) \longrightarrow H_0(C)$$

is dual to $(\phi)_* : H^0(C) \longrightarrow H_3(C)$ and both are isomorphisms. This is easily checked and the result follows since the desired symmetric structure on C_* restricts to the class of the diagonal map in degree 0.

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Remark B.6 In this case the class of the lifting in B.4 determined the class. In general this is not true as there may be many classes in $H_n(\mathbb{Z}\otimes_{\mathbb{Z}(\pi)}(C_*\otimes C_*))$ which lift to the same class in $H_n(C_*\otimes C_*)$. (Here "class" means $d_\#, \hat{d}_\#$ and "lift" multiply by the order of π).

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C. The evaluation of the surgery obstruction for I.1

Proposition C.1 Let τ be a finite 2-group, and suppose C_*, D_* are finitely generated free $\mathbb{Z}(\tau)$-module chain complexes, with a chain equivalence $\lambda_\# : C_* \longrightarrow D_*$. If $\mathbb{Z}/2\otimes_{\mathbb{Z}(\tau)}C_*$ and $\mathbb{Z}/2\otimes_{\mathbb{Z}(\tau)}D_*$ both have trivial boundary maps then $\lambda_\#$ is an injection. Moreover, for each i $D_i/\mathrm{im}\lambda_i$ is a finite odd torsion module.

Proof: Since $\lambda_\#$ is a chain isomorphism

$$\lambda_* : H_*(\mathbb{Z}/2\otimes_{\mathbb{Z}(\tau)}C_*) \longrightarrow H_*(\mathbb{Z}/2\otimes_{\mathbb{Z}(\tau)}D_*)$$

is an isomorphism, but since $\mathbb{Z}/2\otimes_{\mathbb{Z}(\tau)}\partial = 0$ in both complexes it follows that

$$\mathbb{Z}/2\otimes\lambda_\# : \mathbb{Z}/2\otimes_{\mathbb{Z}(\tau)}C_i \longrightarrow \mathbb{Z}/2\otimes_{\mathbb{Z}(\tau)}D_i$$

is an isomorphism for each i. Now apply Nakayama's Lemma and C.1 follows.

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Corollary C.2 Let (D,ψ) be the $(4i+2)$-dimensional quadratic Poincaré complex over $\mathbb{Z}$ with Kervaire invariant 1,

$$D_{2i+1} = \mathbb{Z}\oplus\mathbb{Z} \quad , \quad D_j = 0 \text{ for } j \neq 2i+1$$

$$\psi_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : D^{2i+1} = \mathbb{Z}\oplus\mathbb{Z} \longrightarrow D_{2i+1} = \mathbb{Z}\oplus\mathbb{Z} \ ,$$

and consider the product $(4i+5)$-dimensional quadratic complex

C.3 $$(C_*\otimes D, \phi\otimes\psi) \ .$$

The chain equivalence

C.4 $\phi\otimes\psi - \phi^*\otimes\psi^* : (C_*\otimes D)^{4i+5-*} \longrightarrow C_*\otimes D$

is an injection in each degree, with odd torsion cokernel.

Proof: A direct application of C.1.

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In dimension 0

$$\lambda_0 = \phi_0\otimes\psi - \phi^3\otimes\psi^* = \begin{pmatrix} -1+2y-y^3 & -1 \\ 2y-y^3 & -1+2y-y^3 \end{pmatrix},$$

while in dimension 1

$$\lambda_1 = \phi_1\otimes\psi - \phi^2\otimes\psi^* = \begin{pmatrix} x^3+2y-y^3 & x^3 & * & * \\ 2y-y^3 & x^3+2y-y^3 & * & * \\ 0 & 0 & x+1 & 1 \\ 0 & 0 & x & x+1 \end{pmatrix}.$$

Hence the order of an odd torsion quadratic form representing the surgery obstruction of the product $(C_*\otimes D, \phi\otimes\psi)$ is

$$\det(\phi_1\otimes\psi - \phi^2\otimes\psi^*)/\det(\phi_0\otimes\psi - \phi^3\otimes\psi^*) ,$$

where $\det(\theta)$ means the class of θ in $K_1(\mathbb{Q}(\pi))$. Restricting to the five irreducible representations of π we have the table

	Representation	++	+-	-+	--	Q
C.5	λ_0	1	3	1	3	73
	λ_1	9	-3	1	-3	73

Hence the form in question is represented by a torsion module of order 9 at the trivial representation and 0 at all other representations. Since the form is SKEW SYMMETRIC this must be $\mathbb{Z}/3\oplus\mathbb{Z}/3$ with each $\mathbb{Z}/3$ a torsion lagrangian.

But this torsion class exactly represents the image of the class $\langle 3^{++}\rangle$ from $\hat{H}_{ev}(\mathbb{Z}/2,\tilde{K}_0(\mathbb{Z}(\pi)))$ in $L_1^h(\mathbb{Z}(\pi))$ and we have proved that the surgery obstruction for I.1 is non-trivial.

D. The algebraic evaluation of the surgery obstruction for other space forms

Let τ be one of the groups $\mathbb{Z}/2^m$ or

$Q_2n = \{x,y \mid x^{2^{n-2}} = y^2 = (xy)^2\}$, so that τ acts freely preserving orientation on the sphere S^{2m+1} in the first case or S^{4n+1} in the second. Each such action corresponds to a finite chain complex with the chain homotopy type unambiguously specified by the first k-invariant κ_{i+1} of the resulting quotient S^i/τ. (See e.g. [4] for discussion and references).

For these complexes $C(\tau,\kappa_{m+1})$ Proposition B.2 generalizes, and the only change in the statement is

D.1 $$p_*(e\otimes 1 + 1\otimes e) = |\tau|\Lambda \quad .$$

D.1 together with C.1 provide an effective method for determining the obstruction. When $\tau = \mathbb{Z}/2^m$ here is the result.

Proposition D.2 Let $\tau = \mathbb{Z}/2^m$ and suppose $C(\tau,u_{2n+2})$ is the $\mathbb{Z}(\tau)$-module chain complex of dimension $2n+1$

$$\mathbb{Z}(\tau) \xrightarrow{x^u-1} \mathbb{Z}(\tau) \xrightarrow{\Sigma} \cdots \xrightarrow{\Sigma} \mathbb{Z}(\tau) \xrightarrow{x-1} \mathbb{Z}(\tau) \xrightarrow{\Sigma} \mathbb{Z}(\tau) \xrightarrow{x-1} \mathbb{Z}(\tau)$$

with Poincaré duality chain equivalence $\phi: C(\tau,u_{2n+2})^{2n+1-*} \longrightarrow C(\tau,u_{2n+2})$, for some unit u in the ring $\mathbb{Z}/2^n$. Then

a. For $n+1$ odd $\sigma((C(\tau,u_{2n+2}),\phi)\otimes\text{Kervaire}) = 0 \in L^h_{2n+3}(\mathbb{Z}(\tau))$,

b. For $n+1$ even $\sigma((C(\tau,u_{2n+2}),\phi)\otimes\text{Kervaire}) \neq 0 \in L^h_{2n+3}(\mathbb{Z}(\tau))$ and has non-trivial image in $L^p_3(\mathbb{Z}(\tau)) = \mathbb{Z}/2$.

Proof: The geometrically induced ϕ is such that $\phi_0 = \text{id.}$ and $\phi_{2n+1} = (-)^{n+1}.\text{id}$. Hence a suitable ϕ is given by the table

D.3

dimension	ϕ
0	id
1	$1+x+\ldots+x^{m-u-1}$
2	$(m-u)$
3	$-(m-u)x^{-1}$
4	$-(m-u)$
5	$(m-u)x^{-1}$
6	$(m-u)$
7	⋮
⋮	
2n	$(-1)^{n+1}(1+x^{-1}+\ldots+x^{-(m-u-1)})$
2n+1	$(-1)^{n+1}$

.

Moreover (C_*,ϕ) satisfies the conditions of C.1, so

D.4 $$(C_*\otimes\text{Kervaire}, \phi\otimes\psi - (-1)^{n+1}\phi^*\otimes\psi^*)$$

satisfies the hypothesis of C.1 as well. Hence the answers come from evaluating the alternating product of the images in K_1 of the $(\phi_i\otimes\psi - (-1)^{n+1}\phi^*_{2n+1-i}\otimes\psi^*)$ through dimension $i = n$.

This calculation is direct. The matrices which appear are $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in degree 0, $\begin{pmatrix} 0 & 1+x+..+x^{u-1} \\ -(1+x+..+x^{u-1}) & 0 \end{pmatrix}$ in 1, and $u\begin{pmatrix} 1-x^{-1} & 1 \\ -x^{-1} & 1-x^{-1} \end{pmatrix}$ otherwise. The result is alternately

D.5 $$\left\{\frac{u^2(1-x^{-1}+x^{-2})}{(1+x+..+x^{u-1})^2}\right\} \quad n+1 \text{ even}$$
$$\{(1+x+..+x^{u-1})^2\} \quad n+1 \text{ odd} \quad .$$

The odd case clearly gives 0. For the even case we check at the trivial representation and the -1 representation $r_-(x^i) = (-1)^i$ obtaining

D.6 $$\begin{pmatrix} + & - \\ u^2 & 3u^2 \end{pmatrix}$$

which represents the non-trivial element in $L_3^p(\mathbb{Z}/2^m)$.

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