Recent Advances in Topological Manifolds

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Introduction

A topological n-manifold is a Hausdorff space which is locally n-Euclidean (like $\mathbb{R}^n$).

No progress was made in their study (unlike that in PL, differentiable) until 1968 when Kirby, Siebenmann and Wall solved most questions for high-dimensional manifolds (at least as much for PL and differentiable cases).

Question (1) Can compact n-manifolds be triangulated?

Yes, if $n \leq 3$ (Moise 1950's)

Unknown in general.

Kirby - Siebenmann

However, $\exists$ manifolds (dim ≥ 5) which don't have PL structures. (Might still have triangulations in which links of simplexes aren't PL spheres)

$\exists$ machinery for deciding whether manifold of dim ≥ 5 has a PL structure. (obstruction gp. - $H^4(M; \mathbb{Z}_2)$)

4-manifolds quite unknown in topological, differentiable and PL cases.

(2) ~~Annulus conjecture~~ Generalised Schönflies theorem

Let $B^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| \leq 1\}$ $\quad S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$

Embedding $f : B^n \longrightarrow S^n$ (i.e. 1-1 continuous map)

Is $\overline{S^n \setminus f(B^n)} \cong B^n$? No - Alexander horned sphere.

(True for PL)

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Let $\lambda B^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| \leq \lambda\}$

(2') Is $\overline{S^n \setminus f(\lambda B^n)} \cong B^n$? (where $0 < \lambda < 1$)

Yes.

In 1960, Morton Brown, Mazur & Morse proved the following:

If $g : S^{n-1} \times [-1,1] \longrightarrow S^n$ is an embedding, then $S^n \setminus g(S^{n-1} \times 0)$ has 2 components D_1, D_2 such that

$$\overline{D_1} \cong \overline{D_2} \cong B^n,$$

which implies (2') as a corollary. (The proof is easier than that of PL topology).

(3) Annulus conjecture

Let $f : B^n \longrightarrow \operatorname{Int} B^n$ be an embedding.

Is $\overline{B^n \setminus f(\frac{1}{2}B^n)} \cong \overline{B^n \setminus \frac{1}{2}B^n}$ ($\cong S^{n-1} \times I$)?

In 1968, Kirby, Siebenmann and Wall proved this for $n \geq 5$. (Already known for n...

$n = 4$ still unknown.

Outline of course:

Basic facts about topological manifolds

Morton Brown's theorem - first 'recent' result

Kirby's trick — Homeo(M) is a topological group (compact-open topology).

This is locally contractible: any homeo h near 1 can be joined by a path in Homeo(M) to 1

Product structure theorem: If M^n is a topological manifold, $M \times \mathbb{R}^k$ has a PL structure, then M^n has PL structure. ($n \geq 5$)

Sketch of proof of annulus conjecture (complete except for deep PL theorems) (Also ...

§1. Basic properties of topological manifolds

Let $\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$

Identify $\mathbb{R}^{n-1}$ with $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n = 0\} = \partial\mathbb{R}^n_+$.

Definition 1.1 A (topological) n-manifold (with boundary) is a Hausdorff space M such that each point of M has a neighbourhood homeomorphic to $\mathbb{R}^n_+$. The interior of M, $\operatorname{Int} M$, is the set of points in M which have neighbourhoods homeomorphic to $\mathbb{R}^n$. The boundary of M, $\partial M = M \setminus \operatorname{Int} M$.

$\operatorname{Int} M$ is an open set in M, ∂M is closed set in M.

M is an open manifold if it is non-compact and $\partial M = \emptyset$

M is a closed manifold if it is compact and $\partial M = \emptyset$.

Examples (1) Any open subset of an n-manifold is an n-manifold

(2) Let M be a connected manifold with $\partial M = \emptyset$. If $x, y \in M$ then $\exists$ homeomorphism $h: M \to M$ with $h(x) = y$.

Theorem 1.2 (Invariance of domain) Let $U, V \subset \mathbb{R}^n$ be subsets such that $U \cong V$. Then if U is open in $\mathbb{R}^n$, then so is V.

Proof: later (7).

Corollary 1.3 If M is an n-manifold, then ∂M is an (n-1)-manifold without boundary.

Proof. Suppose $x \in M$ and $f: \mathbb{R}^n_+ \to M$ be a homeo onto a neighbourhood N of x in M. Then $x \in \partial M \Leftrightarrow x \in f(\mathbb{R}^{n-1})$ $\quad (*)$

(A) $x \notin f(\mathbb{R}^{n-1}) \Rightarrow x \in f(\mathbb{R}^n_+ \setminus \mathbb{R}^{n-1}) \cong \mathbb{R}^n$

$\Rightarrow x \in \operatorname{Int} M \Rightarrow x \notin \partial M$. $\square$(A)

(B) $x \notin \partial M \Rightarrow x \in \operatorname{Int} M$, i.e. $\exists$ nbhd U of x homeo to $\mathbb{R}^n \subseteq f(\mathbb{R}^n_+)$

so $\exists$ nbhd V of x, which is open in M, s.t. $V \subset U$, homeo to open set in $\mathbb{R}^n$.

$\therefore f^{-1}(V) \subset \mathbb{R}^n_+ \subset \mathbb{R}^n$

By Theorem 1.2, $f^{-1}(V)$ is open in $\mathbb{R}^n$

~~impossible~~.

Suppose $x \notin f(\mathbb{R}^{n-1})$: then $f^{-1}(x) \in \mathbb{R}^{n-1}$, but then $f^{-1}(V)$ can't be a neighbourhood of $f^{-1}(x)$, so $f^{-1}(V)$ not open #.

$\therefore x \in f(\mathbb{R}^{n-1}) \Longrightarrow x \in \partial M$.

$\square$(B)

Now suppose $y \in \partial M$. Let $g: \mathbb{R}^n_+ \to M$ be a homeo onto a nbhd P of y in M. P contains open nhd W of y in M.

Now $W \cap \partial M = W \cap g(\mathbb{R}^{n-1})$ by $(*)$

and $x \in W \cap \partial M \Rightarrow x \in W \cap g(\mathbb{R}^{n-1})$

$\therefore W \cap g(\mathbb{R}^{n-1})$ is a neighbourhood of y in ∂M homeo to open set in $\mathbb{R}^{n-1}$

$\therefore y$ has a nhd. in ∂M homeo to $\mathbb{R}^{n-1}$, as required.

$\square$

Corollary 1.4 If M^m, N^n are manifolds then $M \times N$ is an (m+n)-manifold with $\partial(M \times N) = (\partial M \times N) \cup (M \times \partial N)$,

i.e. $\operatorname{Int}(M \times N) = \operatorname{Int} M \times \operatorname{Int} N$.

Proof: If $x \in M \times N$, then x has a neighbourhood homeomorphic to $\mathbb{R}^m_+ \times \mathbb{R}^n_+ \cong \mathbb{R}^{m+n}_+$, so $M \times N$ is an (m+n)-manifold.

Clearly, $\operatorname{Int} M \times \operatorname{Int} N \subset \operatorname{Int}(M \times N)$.

If $x \in (\partial M \times N) \cup (M \times \partial N)$, then x has neighbourhood homeo to $\mathbb{R}^m_+ \times \mathbb{R}^n$, $\mathbb{R}^m \times \mathbb{R}^n_+$ or $\mathbb{R}^m_+ \times \mathbb{R}^n_+$ - all homeo to $\mathbb{R}^{m+n}_+$ by a homeomorphism carrying x to $\mathbb{R}^{m+n-1}$. By $(*)$, $x \in \partial(M \times N)$

Hence result.

$\square$

Examples (of manifolds)

i) $\mathbb{R}^m$ is an m-manifold without boundary – open
ii) S^m is an m-manifold (stereographic projection gives nhds.) – closed
iii) B^m is a compact m manifold, with boundary S^{m-1}.
iv) $\mathbb{R}^m_+$ is an m-manifold with boundary $\mathbb{R}^{m-1}$.
v) Products of these
vi) $\mathbb{C}P^n$, orthogonal groups $O(n)$ are manifolds.

These are all differentiable manifolds. $\exists$ topological manifolds which do not possess a differentiable structure.

Lemma 1.5 If $X \subset S^n$ is homeomorphic to B^k, then $\tilde{H}_r(S^n \setminus X) = 0 \ \forall r \in \mathbb{Z}$.

Proof: by induction on k.
True if $k=0$: $S^n \setminus \{pt.\} \cong \mathbb{R}^n$
Assume true if $k=l$; we prove it for $k=l+1$.
Choose homeomorphism $f : \underbrace{B^l \times I}_{\cong B^{l+1}} \to X$
Suppose $\alpha \in \tilde{H}_r(S^n \setminus X)$
Take $t \in I$: by induction hypothesis, $\tilde{H}_r(S^n \setminus f(B^l \times t)) = 0$
$\therefore$ α is represented by the boundary of some singular chain c lying in $S^n \setminus f(B^l t)$
$\exists$ nbhd N_t of $f(B^l \times t)$ in S^n such that c lies in $S^n \setminus N_t$.
$\therefore$ $\exists$ open interval $J_t \subset I$ containing t such that c lies in $S^n \setminus f(B^l \times J_t)$.
Since unit interval is compact, we can cover by finitely many of the J_t's.
$\therefore$ $\exists$ dissection $0 = t_0 < t_1 < \ldots < t_k = 1$ s.t. $[t_{p-1}, t_p] \subset$ some J_t.
Let $\phi_{p,q} : \tilde{H}_r(S^n \setminus X) \xrightarrow{inc_*} \tilde{H}_r(S^n \setminus f(B^l \times [t_p, t_q]))$ (where $p < q$).
Now $\phi_{p-1,p}(\alpha) = 0 \ \forall p$.
Suppose inductively that $\phi_{0,i}(\alpha) = 0$ starts with $i=1$.
By main inductive hypothesis, $\tilde{H}_s(S^n \setminus f(B^l \times t_i)) = 0 \quad s = r, r+1$.
Sets $S^n \setminus f(B^l \times [t_p, t_q])$ are open

We have lattice

$$\begin{tikzcd}[column sep=small]
 & S^n \setminus f(B^l \times t_i) & \\
S^n \setminus f(B^l \times [0,t_i]) \arrow[ur, hook] & & S^n \setminus f(B^l \times [t_i, t_{i+1}]) \arrow[ul, hook'] \\
 & S^n \setminus f(B^l \times [0, t_{i+1}]) \arrow[ul, hook] \arrow[ur, hook'] &
\end{tikzcd}$$

Mayer-Vietoris sequence:

$$0 \to \tilde{H}_r(S^n \setminus f(B^l \times [0, t_{i+1}])) \to \tilde{H}_r(S^n \setminus f(B^l \times [0,t_i])) \oplus \tilde{H}_r(S^n \setminus f(B^l \times [t_i, t_{i+1}])) \to 0$$

map induced by inclusion.
Since $\phi_{0,i}(\alpha) = 0$ and $\phi_{i,i+1}(\alpha) = 0$, we have $\phi_{0,i+1}(\alpha) = 0$ · completes small induction.
$\therefore$ $\phi_{0,k}(\alpha) = 0$, i.e. $\alpha = 0$ and $\tilde{H}_r(S^n \setminus X) = 0$, as required. $\square$

Lemma 1.6 If $X \subset S^n$ is homeomorphic to S^k, then $\tilde{H}_r(S^n \setminus X) \cong \tilde{H}_r(S^{n-k-1})$ $\left(= \begin{matrix} \mathbb{Z} & r = n-k-1 \\ 0 & \text{otherwise} \end{matrix}\right)$

Proof: Induction on k. Result is true if $k=0$, for $S^k \setminus$ pair of points $\cong S^{n-1}$.
Assume result holds for $k = l-1$: prove it for $k=l$
Choose homeomorphism $f : S^l \to X$.
Let D_1, D_2 be northern & southern hemispheres of S^l, so $D_1 \cup D_2 = S^l$, $D_1 \cap D_2 \cong S^{l-1}$
Sets $S^n \setminus X$, $S^n \setminus f(D_i)$, $S^n \setminus f(D_1 \cap D_2)$ open. We have lattice

$$\begin{tikzcd}[column sep=small]
 & S^n \setminus f(D_1 \cap D_2) & \\
S^n \setminus f(D_1) \arrow[ur, hook] & & S^n \setminus f(D_2) \arrow[ul, hook'] \\
 & S^n \setminus X \arrow[ul, hook] \arrow[ur, hook'] &
\end{tikzcd}$$

Mayer-Vietoris sequence

$$0 \to \tilde{H}_{r+1}(S^n \setminus f(D_1 \cap D_2)) \to \tilde{H}_r(S^n \setminus X) \to 0$$

since $\tilde{H}_{r+1}(S^n \setminus f(D_1)) \cong \tilde{H}_{r+1}(S^n \setminus f(D_2)) \cong 0$ by previous lemma.
Whence result, by inductive hypothesis. $\square$

Corollary 1.7 If $f : S^{n-1} \to S^n$ is 1-1 and continuous, then $S^n \setminus f(S^{n-1})$ has just 2 components.

Proof: By 1.6, $\tilde{H}_0(S^n \setminus f(S^{n-1})) \cong \tilde{H}_0(S^0) \cong \mathbb{Z}$

$\therefore$ $S^n \setminus f(S^{n-1})$ has two components.

$\square$

Corollary 1.8 If $f : B^n \to S^n$ is 1-1 and continuous then $f(\text{Int } B^n)$ is open in S^n.

Proof: By Lemma 1.5 $\tilde{H}_0(S^n \setminus f(B^n)) = 0$, so $S^n \setminus f(B^n)$ is connected.

Now $S^n \setminus f(S^{n-1}) = f(\text{Int } B^n) \cup S^n \setminus f(B^n)$

and $f(\text{Int } B^n)$, $S^n \setminus f(B^n)$ are connected, ~~and~~ while $S^n \setminus f(S^{n-1})$ is not (by Corollary 1.7). Thus $f(\text{Int } B^n)$ and $S^n \setminus f(B^n)$ are the components of $S^n \setminus f(S^{n-1})$, and are closed in $S^n \setminus f(S^{n-1})$.

Now ~~$S^n \setminus X$~~ $f(\text{Int } B^n)$ open in $S^n \setminus f(S^{n-1})$, therefore open in S^n

$\square$

Proof of theorem 1.2:

We have $U, V \subset \mathbb{R}^n$, homeo $f : U \to V$, U open in $\mathbb{R}^n$.

Choose $x \in U$. Then $\exists$ closed n-ball $B^n \subset U$ with centre x.

$\exists$ map $g : \mathbb{R}^n \to S^n$ which is homeo onto $g(\mathbb{R}^n)$ (e.g. inverse of stereographic projection)

$gf : U \to S^n$ is 1-1 and continuous, so by 1.7 $gf(\text{Int } B^n)$ is open in S^n, and $f(B^n)$ is open in $\mathbb{R}^n$.

Now $f(x) \in f(\text{Int } B^n) \subset f(U) = V$, so V is a neighbourhood of $f(x)$.

Since $V = f(U)$, V is open in $\mathbb{R}^n$.

$\square$

§2. The generalized Schönflies theorem

Defn 2.1 If M, N are manifolds, an embedding of M in N is a map $f : M \to N$ which is a homeomorphism onto $f(M)$.

[If M is compact then any 1-1 continuous map $f : M \to N$ is an embedding, but this is not true in general].

Theorem 2.2 (Morton Brown's Schönflies theorem)

If $f : S^{n-1} \times [-1,1] \to S^n$ is an embedding, then each component of $S^n \setminus f(S^{n-1} \times 0)$ has closure homeomorphic to B^n.

Proof: below.

Definition 2.3 Let M be a manifold and $X \subset \text{Int } M$.

X is cellular if it is closed and, for any open set U containing

$\exists$ set $Y \subset U$ such that $Y \cong B^n$ and $X \subset \text{Int } Y$.

Examples i) Any collapsible polyhedron in $\mathbb{R}^n$ is cellular

ii) If $f : B^n \to S^n$ is any embedding, then $\overline{S^n \setminus f(B^n)}$ is cellular.

Lemma 2.4 If M is a manifold, and $X \subset M$ is cellular then $M/X \cong M$

by a homeomorphism fixed on ∂M.

Proof: Since X cellular, $\exists\, Y_0 \subset \text{Int } M$, $Y_0 \cong B^n$ and $X \subset \text{Int } Y_0$.

Y_0 has metric d. Let $U_r = \{ y \in Y_0 \mid d(X, y) < \frac{1}{r} \}$. Define Y_r inductively: assume $Y_{r-1} \subset M$ constructed with $X \subset \text{Int } Y_{r-1}$.

X cellular $\Rightarrow \exists\, Y_r \subset (\text{Int } Y_{r-1}) \cap U_r$ such that $Y_r \cong B^n$ and $X \subset \text{Int }$

[$\text{Int } Y_r$ = interior of Y_r in M]

$$Y_0 \supset \text{Int } Y_0 \supset Y_1 \supset \text{Int } Y_1 \supset \dots \supset X = \bigcap_{r=0}^{\infty} Y_r$$

(9) We construct homeomorphisms $h_r : M \to M$ such that

i) $h_0 = 1$

ii) $h_r|_{M\setminus Y_{r-1}} = h_{r-1}|_{M\setminus Y_{r-1}}$

iii) $h_r(Y_r)$ has diameter $< \frac{1}{r}$ (w.r.t. metric d)

Suppose h_{r-1} is defined. Choose a homeomorphism

$$f : h_{r-1}(Y_{r-1}) \to B^n.$$

Now $Y_r \subset \operatorname{Int} Y_{r-1}$, so $f(h_{r-1}(Y_r)) \subset \operatorname{Int} B^n$ and $\exists \lambda < 1$ s.t. $f(h_{r-1}(Y_r)) \subset \lambda B^n$, and also $\exists \varepsilon > 0$ s.t. $F^{-1}(\varepsilon B^n)$ has diameter $< \frac{1}{r}$.

$\exists$ homeomorphism $g : B^n \to B^n$ such that $g|_{\partial B^n} = 1$ and $g(\lambda B^n) \subset \varepsilon B^n$

Define $h_r : M \to M$ by

$$h_r(x) = \begin{cases} h_{r-1}(x) & x \in M\setminus Y_{r-1} \\ f^{-1} g f h_{r-1}(x) & x \in Y_{r-1} \end{cases}$$

To verify iii):

$$h_r(Y_r) \subseteq f^{-1} g f h_{r-1}(Y_{r-1}) \subset f^{-1} g(\lambda B^n) \subset f^{-1}(\varepsilon B^n)$$

has diameter $< 1/r$.

Define $h(x) = \lim_{r\to\infty} h_r(x)$ for each $x \in M$.

If $x \in M\setminus X$ then $x \in M\setminus Y_r$ for some r, and

$$h_r(x) = h_{r+1}(x) = \ldots = h(x)$$ by ii), so $h(x)$ exists.

Since $h_r(Y_r) \supset h_{r+1}(Y_r) \supset \ldots$, with diam $h_r(Y_r) \to 0$

$\bigcap_{r=1}^{\infty} h_r(Y_r) = \{y\}$ for some $y \in M$.

If $x \in X$, $h_r(x) \in h_r(Y_r)$ so $d(h_r(x), y) < \frac{1}{r}$ by iii), so $h_r(x) \to y$ as $r \to \infty$ and $h(x) = y$.

(10) h is continuous at $x \in M\setminus X$ because $h = h_r$ in a neighbourhood of x (for some r).

h is continuous at $x \in X$ because Y_r is a neighbourhood of x and $h(Y_r) \subset \frac{1}{r}$ nhd. of Y. ~~Thus h induces a~~

Thus h induces a continuous map

$$\hat{h} : M/X \to M \qquad \hat{h}|_{\partial M} = 1.$$

Since h coincides with some h_r outside X

$h|_{M\setminus X} \to M\setminus\{y\}$ is a homeomorphism

$h(X) = y$, so $\hat{h}$ is bijective.

$\hat{h}|_{M\setminus X}$ is open

If U is a nbhd. of X in M then $U \supset Y_r$ some r,

so $y \in h_{r+1}(Y_{r+1}) \subset \operatorname{Int} h_r(Y_r) \subset h(U)$

and $h(U)$ is a neighbourhood of y, so h is open.

$\therefore$ $\hat{h}$ is a homeomorphism.

$\square$

<u>Lemma 2.5</u> If $X \subset \operatorname{Int} B^n$ is closed and B^n/X is homeo to some subset of S^n, then X is cellular.

<u>Proof</u>: Let $f : B^n \to S^n$ induce an embedding $\hat{f} : B^n/X \to S^n$

suppose $f(X) = y$.

Then $f(B^n) = \hat{f}(B^n/X) \neq S^n$. (Apply Thm 1.2 to nhds. of pts. of ∂B

Let U be any neighbourhood of X in B^n; $f(U)$ is a neighbourhood of y in S^n.

(11) $f(B^n)$ is a proper closed subset of S^n.

$\exists$ homeomorphism $h: S^n \to S^n$ such that $h|_{\text{some nhd. of } y} = 1$ and $h(f(B^n)) \subset f(U)$:

(For $\exists Y \subset S^n$ s.t. $Y \cong B^n$, $f(B^n) \subset \operatorname{Int} Y$.

Let Z be a small convex ball with $y \in \operatorname{Int} Z$. Then radial map gives homeo.)

Define $g: B^n \to B^n$ by

$$g(x) = \begin{cases} f^{-1} h f(x) & x \notin X \\ x & x \in X \end{cases} \qquad (hf(x) \neq y \Rightarrow f^{-1} h f x \text{ well-defined})$$

Then g is continuous since $h = 1$ in nhd. of y. Also g is 1-1.

Now $g(B^n) \cong B^n$ and $g(B^n) \subset f^{-1} h f(B^n) \subset f^{-1} f(U) = U$, and $g = 1$ on a nhd. of X. Therefore $\operatorname{Int} g(B^n) \supset X$, and X is cellular

$\square$

<u>Proof of Theorem 2.2</u> $f: S^{n-1} \times [-1,1] \to S^n$ is embedding

$S^n \setminus f(S^{n-1} \times 0)$ has two components, D_+ and D_-.

Say $f(S^{n-1} \times -1) \subset D_-$.

Let $X_+ = D_+ \setminus f(S^{n-1} \times (0,1))$

$X_- = D_- \setminus f(S^{n-1} \times (-1,0))$.

Then X_+ and X_- are both closed, and $X_+ \cup X_- = S^n \setminus f(S^{n-1} \times (-1,1))$

Note that $(S^n/X_+)/X_- \cong \left(\dfrac{S^{n-1} \times [-1,1]}{S^{n-1} \times -1}\right) \Big/ S^{n-1} \times 1 \cong S^n$

$\therefore \exists$ map $g: S^n \to S^n$ s.t. $g(X_+) = y_+$, $g(X_-) = y_-$

and $g|_{S^n \setminus (X_+ \cup X_-)}$ is a homeomorphism onto $S^n \setminus \{y_+, y_-\}$

(12) y_+, y_- are the poles of S^n.

$X_+ \cup X_-$ is a proper closed subset of S^n

$\therefore \exists Y \subset S^n$, $Y \cong B^n$, $X_+ \cup X_- \subset \operatorname{Int} Y$.

Since $g(Y)$ is a proper closed subset of S^n, $\exists$ homeomorphism $h: S^n \to S^n$, such that $h = 1$ on a neighbourhood of y_- and $h(g(Y)) \subset S^n \setminus \{y_+\}$

Define $\phi: Y \to S^n$ by

$$\phi(x) = \begin{cases} g^{-1} h g(x) & x \notin X_- \\ x & x \in X_- \end{cases}$$

Continuous since $h = 1$ on a neighbourhood of y_-

ϕ is injective on $Y \setminus X_+$ and $\phi(X_+) = g^{-1} h(y_+)$

$\therefore \phi$ induces an embedding $\hat{\phi}: Y/X_+ \to S^n$, $Y \cong B^n$

By Lemma 2.5, X_+ is cellular

$\overline{D_+}$ is a manifold with $X_+ \subset D_+ = \operatorname{Int} \overline{D_+}$

By Lemma 2.4, $\overline{D_+} \cong \overline{D_+}/X_+ \cong \dfrac{S^{n-1} \times [0,1]}{S^{n-1} \times 1} \cong B^n$

Similarly for D_-.

$\square$

<u>Corollary 2.6</u> If $f, g: S^{n-1} \times [-1,1] \to S^n$ are embeddings, then $\exists$ homeomorphism $h: S^n \to S^n$

(13)

$$hf|_{S^{n-1}\times 0} = g|_{S^{n-1}\times 0}$$

Proof: If $\phi : \partial B^n \longrightarrow \partial B^n$ is a homeomorphism, then ϕ extends to a homeomorphism $\phi : B^n \longrightarrow B^n$ (in obvious way along radii : $\phi(rx) = r\phi(x)$ $0 \le r < 1$ $x \in \partial B$)

$\therefore$ If Y_1, Y_2 are homeomorphic to balls and $\phi : \partial Y_1 \to \partial Y_2$ is a homeomorphism, then ϕ extends to a homeomorphism $\phi : Y_1 \longrightarrow Y_2$.

Let D_+, D_- be components of $S^n \setminus f(S^{n-1}\times 0)$

E_+, E_- — " — $S^n \setminus g(S^{n-1}\times 0)$

Define $h|_{f(S^{n-1}\times 0)}$ to be gf^{-1}, so $h : \partial\bar{D}_+ \longrightarrow \partial\bar{E}_+$.

Since $\bar{D}_+ \cong \bar{E}_+ \cong B^n$, h can be extended to a homeomorphism $h : \bar{D}_+ \longrightarrow \bar{E}_+$.

Extend $h|_{\partial\bar{D}_-} \to \partial E_-$ (already defined) to homeo $h|_{\bar{D}_-} \longrightarrow \bar{E}_-$.

Obtain homeo $h : S^n \longrightarrow S^n$ with $hf|_{S^{n-1}\times 0} = g|_{S^{n-1}\times 0}$. □

Definition 2.7 A collar of ∂M in M is an embedding $f : \partial M \times I \longrightarrow M$ such that $f(x, 0) = x$ $(x \in \partial M)$. □

Exercise $f(\partial M \times I)$ is a neighbourhood of ∂M in M.

(14)

Remark: From now on, we only consider metrizable manifolds (i.e. ones which are 2nd countable)

Exercise Compact manifolds are metrizable.

Theorem 2.8 (Morton Brown) If M is metrizable, then ∂M has a collar in M.

□ Proof below

[redacted] If U is an open set in ∂M, say that U is collared if U has a collar in the manifold Int $M \cup U$

Let $V \subset U$ be a smaller open set.

Let $\lambda : U \to I = [0,1]$ be a continuous map such that $\lambda(x) = 0 \Leftrightarrow x \notin V$. Define a spindle neighbourhood of V in $U \times I$ to be

$$S(V, \lambda) = \{(x,t) \in U\times I \mid t < \lambda(x)\}$$ (open: so a neighbourhood of $V \times 0$).

Lemma 2.9 Let $f : S(V,\lambda) \longrightarrow U\times I$ be an embedding with $f|_{V\times 0} = 1$. Then $\exists$ homeo $h : U\times I \longrightarrow U\times I$ such that:

i) $hf = 1$ on $S(V,\mu)$ for some μ s.t. $\mu \le \lambda$

ii) $h|_{U\times I \setminus f(S(V,\lambda))}$ is identity

Proof: Spindle nbhds. form a base of nbhds.

(15) of $V\times 0$ in $U\times I$. Suppose $V\times 0\subset W$, W open.

Let d be a metric on U: define a metric d on $U\times I$ by

$$d((x,t),(x',t')) = d(x,x') + |t-t'|$$

Let $\nu(x) = \min(d(x,0), U\times I\setminus W, d(x, U\setminus V))$

$(x,t)\in S(V,\nu) \Rightarrow t<\nu(x) \Rightarrow (x,t)\in W$

$\therefore S(V,\nu)\subset W$

$\exists \mu$ s.t. $S(V,2\mu)\subset S(V,\tfrac12\lambda)\cap f(S(V,\tfrac12\lambda))$

$\exists$ embedding $g: U\times I \longrightarrow U\times I$; $(x,t)\longmapsto \begin{cases}(x,t) & t\geqslant 2\mu(x)\\ (x,\mu(x)+\frac12 t) & t<2\mu(x)\end{cases}$
with image $U\times I\setminus S(V,\mu)$ and $g|_{U\times I\setminus S(V,2\mu)} = 1$.

Define $h: U\times I\longrightarrow U\times I$ by

$$h(x) = \begin{cases} f^{-1}(x) & x\in fS(V,\mu)\\ gfg^{-1}f^{-1}(x) & x\in fS(V,\lambda)\setminus fS(V,\mu)\\ x & x\notin f(S(V,\lambda/2)).\end{cases}$$

~~h continuous on $t=\mu(x)$ because $gfg^{-1}f^{-1}(x,\mu(x))$~~

Continuity of h is simply verified.
In fact, h is a homeomorphism s.t. $hf=1$ on $S(V,\mu)$ and $h=1$ off $f(S(V,\lambda))$.

$\square$

(16) <u>Lemma 2.10</u> If $U,V\subset\partial M$ are collared, then $U\cup V$ is collared.

<u>Proof</u>: Let $f: U\times I\longrightarrow M$, $g: V\times I\longrightarrow M$ be collar

Choose $\lambda: U\cup V\longrightarrow I$ so that $S(U\cap V,\lambda)\subset f^{-1}g(V\times$

Apply Lemma 2.9 to the embedding

$$g^{-1}\circ f: S(U\cap V,\lambda)\longrightarrow V\times I.$$

$\exists S(U\cap V,\mu)\subset S(U\cap V,\lambda)$ and a homeomorphis

$$h: V\times I\longrightarrow V\times I$$

s.t. $hg^{-1}f|_{S(U\cap V,\mu)} = 1$. Then gh^{-1} and f agree on $S(U\cap V,\mu)$.

Define open set $U_1\subset U\times I$ by

$$U_1 = \{x\in U\times I \mid d(x,(U\setminus V)\times 0) < d(x,(V\setminus U)\times 0)\}$$

Define $V_1\subset V\times I$ similarly: then

~~where~~
Let $U_2 = \{y\in M \mid d(y,U\setminus V) < d(y,V\setminus U)\}$, V_2 similarly

So $U_1\cap V_1=\emptyset$, $U_2\cap V_2=\emptyset$

Put $U_3 = U_1\cap f^{-1}(U_2)$, $V_3 = V_1\cap hg^{-1}(V_2)$

(17) Then U_3, V_3 open, $U_3 \cap V_3 = \emptyset$, $f(U_3) \cap gh^{-1}(V_3) = \emptyset$
$(U \setminus V) \times 0 \subset U_3$, $(V \setminus U) \times 0 \subset V_3$ so
$W = U_3 \cup S(U \cap V, \mu) \cup V_3$ is a neighbourhood of $(U \cup V) \times 0$ in $(U \cup V) \times I$.

Define $\phi : W \longrightarrow M$ by $\phi(x) = \begin{cases} f(x) & x \in U_3 \cup S(U \cap V, \mu) \\ gh^{-1}(x) & x \in S(U \cap V, \mu) \cup V_3 \end{cases}$

Then ϕ is well-defined, ~~and continuous.~~
Continuous and 1-1.

$\exists\, \nu : U \cup V \longrightarrow I$ such that $S(U \cup V, \nu) \subset W$.
Define $\psi : (U \cup V) \times I \longrightarrow M$;

$$(x,t) \longmapsto \phi(x, t\tfrac{\nu(x)}{2})$$

This is continuous and 1-1, and hence an embedding (invariance of domain). $\square$

Proof of Theorem 2.8:

i) collared sets cover ∂M because
$x \in \partial M \Rightarrow \exists$ homeo $f : \mathbb{R}^n \times \mathbb{R}_+ \longrightarrow M$ onto a nbhd of x in M.
We proved in Cor 1.3 $f(\mathbb{R}^n \times 0)$ contains a nbhd U of x in ∂M.
Then U has a collar given by

$$g.\ U \times I \longrightarrow M ;\ (y,t) \longmapsto f(f^{-1}(y), t)$$

(18) If ∂M is compact, then ∂M is collared by Lemma 2.10. Before proceeding to general case we prove:

Lemma 2.10½ Let U_α $(\alpha \in A)$ be a disjoint family of open collared sets. Then $\bigcup_{\alpha \in A} U_\alpha$ is collared.

Proof: Let $V_\alpha = \{y \in M \mid d(y, U_\alpha) < d(y, \bigcup_{\beta \neq \alpha} U_\beta)\}$.
This is an open neighbourhood of U_α in M and

$$\alpha \neq \beta \Rightarrow V_\alpha \cap V_\beta = \emptyset.$$

Let $f_\alpha : U_\alpha \times I \longrightarrow M$ be a collar of U_α.
Let $W_\alpha = f_\alpha^{-1}(V_\alpha)$, a neighbourhood of $U_\alpha \times 0$ in $U_\alpha \times I$.
$\exists\, \nu_\alpha : U_\alpha \longrightarrow I$ such that $S(U_\alpha, \nu_\alpha) \subset W_\alpha$.
Define $g_\alpha : U_\alpha \times I \longrightarrow M$ by

$$g_\alpha(x,t) = f_\alpha(x, t\nu_\alpha(x)/2) \in V_\alpha.$$

Define $g = \bigcup g_\alpha : (\bigcup_\alpha U_\alpha) \times I \longrightarrow M$
This is a collar of $\bigcup_{\alpha \in A} U_\alpha$ in M $\square$

(19)

We have proved that if $X = \partial M$ then

i) X is ~~collared~~ covered by collared sets

ii) finite union of collared sets collared

iii) disjoint union of collared sets is collared

iv) open subsets of collared sets ~~to~~ are collared.

Then i) – iv) + X metric $\Longrightarrow$ X collared.

Lemma 2.10¾ Any countable union of collared sets is collared:

Proof: enough to consider countable nested unions

$$U = \bigcup_{1}^{\infty} U_n \quad \text{with } U_1 \subset U_2 \subset \dots$$

Put $V_n = \{x \in U_n \mid d(x, X\setminus U_n) > 2^{-n}\}$

Then $U = \bigcup_{1}^{\infty} V_n$ for $x \in U_k \Rightarrow \exists n : B(x, 2^{-n}) \subset U_k$

$\Rightarrow d(x, X\setminus U_k) > 2^{-n} \Rightarrow d(x, X\setminus U_n) > 2^{-n} \Rightarrow x \in V_n$.

Now $\overline{V}_n \subset V_{n+1}$.

Let $A_k = V_{2k+1} \setminus \overline{V}_{2k-1}$ $\quad B_k = V_{2k+2} \setminus \overline{V}_{2k}$.

Then $A = \bigcup_{k=1}^{\infty} A_k$ is disjoint union of collared sets – hence collared. Similarly for $B = \bigcup_{k=1}^{\infty} B_k$

Now $U = A \cup B \cup V_2$ is collared.

$\square$

(20)

Call a family of subsets of X <u>σ-disjoint</u> if it is a countable union of disjoint $\left\{\begin{array}{l}\text{unions}\\ \text{families}\end{array}\right.$.

<u>Lemma 2.10⅞</u> Every open cover of a metric space X has a σ-disjoint refinement.

<u>Proof</u> (cf Kelley, p. 129)

Let $\mathcal{U}$ be an open cover of metric space X.

If $U \in \mathcal{U}$ let $U_n = \{x \in U \mid d(x, X\setminus U) > 2^{-n}\}$

Then $d(U_n, X\setminus U_{n+1}) \geq 2^{-(n+1)}$

Well-order $\mathcal{U}$ by relation $<$.

Let $U_n^* = U_n \setminus \bigcup_{V<U} V_{n+1}$

If $U \neq V$ then $U < V$ or $U > V$

$$\Downarrow \qquad\qquad \Downarrow$$

$$V_n^* \subset X\setminus U_{n+1} \qquad U_n^* \subset X \setminus V_{n+1}$$

and in either case $d(U_n^*, V_n^*) \geq 2^{-(n+1)}$.

Let $U_n' =$ (open $2^{-(n+2)}$ neighbourhood of U_n^*) $\subset$ (

$U \neq V \Rightarrow U_n' \cap V_n'$ disjoint

Enough to prove $\bigcup_{n,U} U_n' = X$.

If $x \in X$ let U be <u>first</u> (w.r.t. $<$) member of $\mathcal{U}$ containing x. Then $x \in U_n$ for some n and

(21)

so $x \in U_n^* \subset U_n'$.

Now $\{U_n'\}$ is a σ-disjoint refinement of $\mathcal{U}$.

$\square$

$\square$ Thm 2.8

References:

1) Morton Brown: "A proof of the generalized Schönflies conjecture" Bull. Amer. Math. Soc. 66 (1960) 74-76

2) Morton Brown: "Locally flat embeddings of topological manifolds" Annals of Maths. 75 (1962) 331-341

A shortened version of 2) is included in the book "Topology of 3-manifolds".

<u>Definition 2.11</u> Let M^m, N^n be manifolds without boundary. An embedding $f: M^m \longrightarrow N^n$ is <u>locally flat</u> if, for all $x \in M$, $\exists$ neighbourhood U of x and embedding $F: U \times \mathbb{R}^{n-m} \longrightarrow N^n$ s.t. $F(y,0) = f(y)$ $(y \in U)$. $\square$

N.B. needn't be an embedding $G: M \times \mathbb{R}^{n-m} \longrightarrow N$ s.t. $G(y,0) = f(y)$ $(\forall y)$.

e.g. $S^1 \to$ Möbius strip (along centre line). This is locally flat but $\nexists$ embedding $S^1 \times \mathbb{R} \longrightarrow M$ agreeing with previous one on $S^1 \times 0$.

(22)

<u>Examples</u> 1) If $f: S^{n-1} \longrightarrow S^n$ is locally flat then each component of $S^n \setminus f(S^{n-1})$ has closure homeo to B^n.

2) If ∂M is compact, and $f, g: \partial M \times I \longrightarrow M$ are two collars, then $\exists$ homeomorphism $h: M \longrightarrow M$ such that hf agrees with g on $\partial M \times [0, \frac{1}{2}]$, and $h = 1$ outside $f(\partial M \times I) \cup g(\partial M$

"Collaring of ∂M in M is unique"

Not true if ∂M non-compact; (Milnor's rising sun).

Exercise: suggest a generalization that does work.

Given two manifolds M^m, N^n let $\mathcal{E}(M,N)$ be the set of embeddings of M in N with the compact-open topology.

A cts. map $f: X \longrightarrow Y$ is <u>proper</u> if $C \subseteq Y$ compact $\Rightarrow f^{-1}(C) \subseteq X$ compact.

Let $\mathcal{E}_p(M,N)$ be the set of embeddings which are proper maps.

We shall be interested in $\mathcal{E}_p(\mathbb{R}^n \setminus \operatorname{int} B^n, \mathbb{R}^n)$, which consists of embeddings $f: \mathbb{R}^n \setminus \operatorname{int} B^n \longrightarrow \mathbb{R}^n$ <u>onto</u> nbhds of ∞

(23) Let $\hat{\mathbb{R}}^n$ be the 1-pt. compactification of $\mathbb{R}^n$.

$f : \mathbb{R}^n \setminus \operatorname{int} B^n \longrightarrow \mathbb{R}^n$ extends to a continuous map $\hat{f} : \hat{\mathbb{R}}^n \setminus \operatorname{int} B^n \longrightarrow \hat{\mathbb{R}}^n$ $(\hat{f}(\infty) = \infty)$ iff f is proper.

(In general $f : X \to Y$ extends to a continuous map $\hat{f} : \hat{X} \longrightarrow \hat{Y}$ $(\hat{f}(\infty) = \infty)$ iff f is proper).

Theorem 2.12 There is a neighbourhood U of 1 in $\mathcal{E}(6B^n \setminus \operatorname{int} B^n, \mathbb{R}^n)$ ~~st~~ and a continuous map

$$\theta : U \longrightarrow \mathcal{E}_p(\mathbb{R}^n \setminus \operatorname{int} B^n, \mathbb{R}^n)$$

s.t. $\theta(f)|_{S^{n-1}} = f|_{S^{n-1}}$.

Proof: Take $U = \{ f \in \mathcal{E}(6B^n \setminus \operatorname{int} B^n, \mathbb{R}^n) \mid d(x, f(x)) < 1, \ x \in 6B^n \setminus \operatorname{int} B^n \}$

If $f \in U$, then $f(2B^n \setminus \operatorname{int} B^n) \subseteq \operatorname{int} 3B^n \setminus 0$ and $f(6B^n \setminus \operatorname{int} 5B^n) \subset f(7B^n \setminus \operatorname{int} 4B^n)$.

Define inductively

$$f_k : (4k+6) B^n \setminus \operatorname{int} B^n \longrightarrow \mathbb{R}^n \text{ s.t.}$$

i) $f_0 = f$

ii) $f_{k+1}|_{(4k+5)B^n \setminus \operatorname{int} B^n} = f_k|_{(4k+5)B^n \setminus \operatorname{int} B^n}$

(24) iii) $f_k((4r+6)B^n \setminus \operatorname{int}(4r+5)B^n) \subset \operatorname{int}(4r+7)B^n \setminus (4r+4)B^n$ for $r \leq k$

iv) f_k depends continuously on f.

Suppose f_k constructed.

If $c, d \in (a,b)$ $(a,b,c,d \in \mathbb{R})$ let $p(a,b,c,d) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the radial homeomorphism fixed outside $bB^n \setminus aB^n$, taking cB^n onto dB^n.

Let $\rho_k(a,b,c,d) = p(4k+a, 4k+b, 4k+c, 4k+d)$.

Define $g_k : (4k+6)B^n \setminus \operatorname{int} B^n \longrightarrow \mathbb{R}^n$ by

$$g_k = \rho_k(3,11,4,8)\, f_k\, \rho_k(1, 5\tfrac{2}{3}, 5\tfrac{1}{3}, 2)$$

Define homeo $h_k : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by

$$h_k(x) = \begin{cases} f_k \rho_k(1, 5\tfrac{2}{3}, 2, 5\tfrac{1}{3}) f_k^{-1} & (x \in \text{image } f_k) \\ x & (\text{otherwise}) \end{cases}$$

Let $\sigma_k : (4k+10)B^n \longrightarrow (4k+6)B^n$ be a radial homeo fixed on $(4k+5)B^n$, sending $(4k+6)B^n \longrightarrow (4k+5\tfrac{1}{2})B^n$, $(4k+9)B^n \longrightarrow (4k+5\tfrac{2}{3})B^n$

Define $f_{k+1} = h_k g_k \sigma_k : (4k+10)B^n \setminus \operatorname{int} B^n \longrightarrow \mathbb{R}^n$

Check ii). let $x \in (4k+5)B^n$ so $\sigma_k(x) = x$, $\rho_k(1, 5\tfrac{2}{3}, 5\tfrac{1}{3}, 2)(x) \in (4k+2)B^n$
$y = f_k \rho_k(1, 5\tfrac{2}{3}, 5\tfrac{1}{3}, 2)(x) \in (4k+3)B^n$ (induction hypothesis), $g_k(x) = \rho_k(3,11,4,8)(y) = y$, $f_{k+1}(x) = h_k(y) = f_k(x)$, as reqd

(25) Similarly, we can verify ~~iii~~ iii)

To prove iv), that f_k depends continuously on f, it is enough to show that h_k depends continuously on f_k. Let f'_k be near f_k, and let

$$h'_k = \begin{cases} f'_k \rho_k f_k^{-1} & \text{on } \operatorname{im} f_k \\ 1 & \text{otherwise} \end{cases} \quad \text{where } \rho_k = \rho_k(1, 5\tfrac{2}{3}, 3, 5\tfrac{1}{3})$$

If C is a compact set in $\mathbb{R}^n$, must prove that $\sup_{x\in C} d(h_k x, h'_k x)$ can be made $< \varepsilon$ by requiring $d(f_k y, f'_k y) = \delta$ $(\forall y \in A_k)$

$$f_k \longmapsto h_k \qquad \mathcal{E}(6B^n \setminus \operatorname{Int} B, \mathbb{R}^n) \longrightarrow \operatorname{Homeo} \mathbb{R}^n$$

Let $A_k = (4k+6)B^n \setminus \operatorname{Int} B^n$ = domain of f_k

Given $\varepsilon > 0$ $\exists\, \eta > 0$ s.t. $y, y' \in A$, $d(y,y') < \eta \Rightarrow d(f_k\rho_k(y), f_k\rho_k(y')) < \varepsilon/2$. Since f_k injective, $\exists\, \delta > 0$ s.t. $y, y' \in A$ $d(y,y') \geq \eta \Rightarrow d(f_k y, f_k y') \geq \delta$.

We suppose $\delta < \varepsilon/2$.

Suppose $d(f_k y, f'_k y) < \delta/2$ for all $y \in A$

Let $x \in C$. We split into cases:

(26)

i) $x \in \operatorname{im} f_k \cap \operatorname{im} f'_k$, say $x = f_k y_k = f'_k y'_k$

Then $d(f_k y_k, f_k y'_k) = d(f'_k y'_k, f_k y'_k) < \delta/2 < \delta$

$\therefore d(y, y') < \eta$, so

$$d(h_k x, h'_k x) = d(f_k \rho_k y_k, f'_k \rho_k y'_k)$$

$$\leq d(f_k \rho_k y_k, f_k \rho_k y'_k) + d(f_k \rho_k y'_k, f'_k \rho_k y'_k)$$

$$< \varepsilon/2 + \delta/2 < \varepsilon$$

ii) If $x \in \operatorname{im} f_k \setminus \operatorname{im} f'_k$, say $x = f_k(y)$

then $d(x, f'_k y) < \delta/2$, so $\exists\, z \in \partial A$ s.t. $d(f'_k z, f'_k y) < \delta/2$ and $d(x, f'_k z) < \delta/2$.

But $d(f'_k z, f_k z) < \delta/2$, so $d(x, f_k z) < \delta$, so $d(y,z) < \eta$

$$\therefore d(h_k x, h'_k x) = d(f_k \rho_k y, x) \leq d(f_k \rho_k y, f_k \rho_k z) + \underbrace{d(f_k z, x)}_{}$$

$\rho_k z = z$ (since $z \in \partial A$)

$$< \varepsilon/2 + \delta < \varepsilon.$$

iii) If $x \in \operatorname{im} f'_k \setminus \operatorname{im} f_k$ similar

iv) If $x \notin \operatorname{im} f'_k \cup \operatorname{im} f_k$ nothing to prove.

(27) We have proved $f_k \longmapsto h_k$ continuous.

$f \longmapsto f_{k+1}$ is continuous if $f \longmapsto f_k$ is

(✓ by ii) on (24))

Induction complete.

Define $\theta : U \longrightarrow \mathcal{E}_p(\mathbb{R}^n \setminus \operatorname{Int} B^n, \mathbb{R}^n)$ by

$\theta(f) : \mathbb{R}^n \setminus \operatorname{Int} B^n \longrightarrow \mathbb{R}^n$ by

$$\theta(f)(x) = f_k(x) \quad \begin{pmatrix} k \text{ large} \\ x \in (4k+5) B^n \end{pmatrix}$$

Then $\theta(f)$ is proper (interleaving prop iii) on (24))

and Also $\theta(f)$ is an embedding, so $\theta(f) \in \mathcal{E}_p(\mathbb{R}^n \setminus \operatorname{Int} B^n, \mathbb{R}^n)$

$\theta(f)$ depends continuously on f because f_k agrees with f on $(4k+5)B^n$ and f_k depends continuously on f.

$\square$.

Corollary 2.13 If $0 < \lambda < 1$, then $\exists$ neighbourhood V of $1 \in \mathcal{E}(B^n \setminus \operatorname{Int} \lambda B^n, \mathbb{R}^n)$ and a continuous map $\phi : V \longrightarrow \mathcal{E}(B^n, \mathbb{R}^n)$ s.t. $\forall f$, $\phi(f)|_{S^{n-1}} = f|_{S^{n-1}}$.

Proof $\hat{X}$ = 1-pt. compactification of X

if $g : X \longrightarrow Y$ proper $\Leftrightarrow$ g extends to $\hat{g} : \hat{X} \longrightarrow \hat{Y}$ with $\hat{g}(\infty) = \infty$.

(28) Ex. The mapping $g \mapsto \hat{g}$ is not a continuous map, even if $X = Y = \mathbb{R}^n$

Proof (of Cor 2.13 contd.)

We first prove ~~that~~ $f \longmapsto \hat{f}$ is continuous

$$\mathcal{E}_p(\mathbb{R}^n \setminus \operatorname{Int} B^n, \mathbb{R}^n) \longrightarrow \mathcal{E}(\hat{\mathbb{R}}^n \setminus \operatorname{Int} B^n, \hat{\mathbb{R}}^n)$$

Suppose $f \in \mathcal{E}_p(\;\;)$, $C \subset \hat{\mathbb{R}}$ cpt. $U \subset \hat{\mathbb{R}}^n$ ~~given~~ open

$\hat{f}(C) \subset U$

If $\infty \notin C$, C is a compact set in $\mathbb{R}^n \setminus \operatorname{Int} B^n$

so $\{ g \in \mathcal{E}_p(\mathbb{R}^n \setminus \operatorname{Int} B^n, \mathbb{R}^n) \mid g(C) \subset U \cap \mathbb{R}^n \}$ is a neighbourhood of f, mapping into given neighbourhood of $\hat{f}$.

If $\infty \in C$, then $\infty = \hat{f}(\infty) \in U$ open in $\hat{\mathbb{R}}^n$

$\exists k$ such that $\hat{\mathbb{R}}^n \setminus k B^n \subset U$

Since f proper, $\exists\, l$ s.t. $f^{-1}(2kB^n) \subset l B^n$

Let $N = \{ g \in \mathcal{E}_p(\;) \mid g(C \cap lB^n) \subset U \cap \mathbb{R}^n$
$g(l S^{n-1}) \subset \mathbb{R}^n \setminus k B^n \}$

This is open in $\mathcal{E}_p$, contains f..

(29)

Have to show $\hat{g}(C) \subset U$ for all $g \in N$

$$\hat{g}(C) = \hat{g}(C \cap lB^n) \cup \hat{g}(\hat{\mathbb{R}}^n \setminus \mathrm{int}\, \ell B^n)$$

$\subset$ $U \cup$ one of complementary domains of $g(\ell S^{n-1})$

in fact $U \cup$ outside domain $\subset \hat{\mathbb{R}}^n \setminus kB^n$

$\subset U$

Hence the map $g \mapsto \hat{g}$ is continuous.

$$\begin{array}{ccc} \mathcal{E}(6B^n \setminus \mathrm{int}\, B^n, \mathbb{R}^n) \supset U & \longrightarrow & \mathcal{E}_p(\mathbb{R}^n \setminus \mathrm{int}\, B^n, \mathbb{R}^n) \\ & \searrow & \downarrow \\ & & \mathcal{E}_p(\hat{\mathbb{R}}^n \setminus \mathrm{int}\, B^n, \hat{\mathbb{R}}^n) \end{array}$$

$\exists$ homeomorphisms $h : \hat{\mathbb{R}}^n \longrightarrow \hat{\mathbb{R}}^n$;

$$h(x) = \begin{cases} x/\|x\|^2 & x \neq 0, \infty \\ \infty & x = 0 \\ 0 & x = \infty \end{cases}$$

carries $6B^n \setminus \mathrm{int}\, B^n$ onto $B^n \setminus \mathrm{int}\, \frac{1}{6} B^n$

~~Let~~ taking $\hat{\mathbb{R}}^n \setminus \mathrm{int}\, B^n \longrightarrow B^n$.

Hence result

$\square$

(30)

§3. Properties of Tori

Definition 3.1 Let $\mathbb{Z}^n$ be integer lattice in $\mathbb{R}^n$. Then $T^n = \mathbb{R}^n / \mathbb{Z}^n$ is the <u>n-dimensional torus</u> (Clearly $T^n \cong \underbrace{S^1 \times \dots \times S^1}_{n \text{ times}}$) $\square$

Let $e : \mathbb{R}^n \twoheadrightarrow T^n$ be projection map.

If $a \in \mathbb{Z}^n$, let $\tau_a : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$; $x \mapsto a + x$.

Proposition 3.2 $e : \mathbb{R}^n \to T^n$ is a universal covering of T^n. If X is a ~~simply~~ connected space and $f : X \to T^n$ is any map then $\exists \tilde{f} : X \to \mathbb{R}^n$ such that $f = e\tilde{f}$. ($\tilde{f}$ is a <u>lift</u> of f).

If $\tilde{f}_1, \tilde{f}_2$ are lifts of f then $\tilde{f}_1 = \tau_a \tilde{f}_2$ for some $a \in \mathbb{Z}^n$.

$\square$

(31)

If X is simply connected $f: X\times T^n \to X\times T^n$ is a map, then $\exists \tilde{f}: X\times\mathbb{R}^n \to X\times\mathbb{R}^n$ such that $e\tilde{f} = fe$.

Lemma 3.3 If f is a homeomorphism so is $\tilde{f}$; if $f \simeq$ identity, then $\tilde{f}$ commutes with the covering translations. $\square$

Proof: f homeo, inverse g. Form $\tilde{f}, \tilde{g}$.

$e\tilde{f}\tilde{g} = fe\tilde{g} = fge = e$

$\therefore \tilde{f}\tilde{g} = \tau_a$ for some a. Similarly $\tilde{g}\tilde{f} = \tau_b$.

$\therefore \tilde{f}$ is a homeo.

Suppose $F: X\times T^n\times I \to X\times T^n$ has $F_0 = f$, $F_1 = 1$. By 3.2 $\exists \tilde{F}: X\times\mathbb{R}^n\times I \to X\times\mathbb{R}^n$ with $e\tilde{F} = Fe$.

Let τ_a be covering translation ~~s.t.~~.

$e\tau_{-a}\tilde{F}\tau_a = e\tilde{F}\tau_a = Fe\tau_a = Fe = e\tilde{F}$.

(32)

$\therefore \exists\, b \in \mathbb{Z}^n$. $\tau_{-a}\tilde{F}\tau_a = \tau_b\tilde{F}$

$e\tilde{F}_1 = F_1 e = e$, so $\tilde{F}_1 = \tau_c$ for some c.

But $\tau_{-a}\tau_c\tau_a = \tau_b\tau_c$

$\therefore b = 0, \tau_b = 1$

$\tau_{-a}\tilde{F}\tau_a = \tilde{F}$

Since $F_0 = f$, $\tilde{F}_0 = \tau_d\tilde{f}$ (for some d).

$\therefore \tilde{f}$ commutes with τ_d.

$\square$

Definition 3.4 Let M, N be manifolds. An *immersion* $f: M \to N$ is a map such that each point $x \in M$ has a neighbourhood U_x with $f|_{U_x}$ an embedding. If U_x can be chosen so that $f|_{U_x}$ is *locally flat* (defined on p. 21) then f is *locally flat immersion*.

(33)

Theorem 3.5 There is an immersion of $(T^n \setminus \text{point})$ in $\mathbb{R}^n$.

Proof: (1) $T^n \setminus pt$ is open parallelizable manifold

∴ By Hirsch's theory of immersions $\exists\, C^\infty$ immersion $T^n \setminus pt \to \mathbb{R}^n$

Alternatively:

(2) Regard T^n as the product of n circles. $T = T^1 =$ circle.
Let J be a closed interval in T.

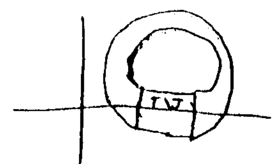
$T^n \setminus J^n \underset{\text{homeo}}{\cong} T^n \setminus pt$. (all open manifolds)

$$(T^n \setminus (2J)^n \supset T^n \setminus J^n \xrightarrow{\text{radial}} T^n \setminus pt)$$

Assume inductively that $\exists$ immersion $f_n : T^n \setminus J^n \longrightarrow \mathbb{R}^n$ such that $f_n \times 1 : (T^n \setminus J^n) \times [-1,1] \longrightarrow \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$ extends to an immersion $g_n : T^n \times [-1,1] \longrightarrow \mathbb{R}^{n+1}$.

Induction starts with $n=1$.

(in fact we can find embeddings)



Let $\phi_0 : \overline{T \setminus J} \longrightarrow [-1,1]$ be a homeo

Choose embedding $\phi : \mathbb{R} \times T \longrightarrow \mathbb{R} \times \mathbb{R}$ s.t. if $(x,t) \in [-1,1] \times \overline{T \setminus J}$ then $\phi(x,t) = (x, \phi_0(t))$.

Extend $\phi_0^{-1} : [-1,1] \longrightarrow \overline{T \setminus J}$ to an embedding $\psi : \mathbb{R} \to T$.

(34) Suppose f_n, g_n constructed

$$T^{n+1} \setminus J^{n+1} = (T^n \setminus J^n) \times T \cup T^n \times (T \setminus J)$$

Define $f'_{n+1} : T^{n+1} \setminus J^{n+1} \longrightarrow \mathbb{R}^{n+1}$ by

$$\cancel{f_{n+1} = (1_{\mathbb{R}^{n-1}} \times \phi)(f_n \times 1_T)}$$

$$f'_{n+1} = (1_{\mathbb{R}^{n-1}} \times \phi)\left[(f_n \times 1_T) \cup (1 \times \psi)\, g_n\, (1_{T^n} \times \psi^{-1})\right]$$

On $(T^n \setminus J^n) \times (T \setminus J)$, $g_n = f_n \times 1$

so $(1 \times \psi) g_n (1 \times \psi^{-1}) = (1 \times \psi)(f_n \times 1)(1 \times \psi^{-1}) = f_n \times 1$

Let $J' = T \setminus \psi(-\frac{1}{4}, \frac{1}{4})$, so $J \subset \operatorname{Int} J'$

We shall construct immersion $g_{n+1} : T^{n+1} \times [-1,1] \to \mathbb{R}^{n+2}$ which agrees with $f'_{n+1} \times I$ on $T^{n+1} \setminus (J')^{n+1} \times [-\frac{1}{4}, \frac{1}{4}]$.

This will be enough, since $T^{n+1} \setminus (J')^{n+1} \cong T^{n+1} \setminus J^{n+1}$

Define $\theta_t : \mathbb{C} \longrightarrow \mathbb{C}\ (= \mathbb{R}^2)$ by

$$\theta_t(z) = \begin{cases} z & |z| \leq 1/2 \\ z e^{2(|z| - \frac{1}{2})\pi i t} & \frac{1}{2} \leq |z| \leq 3/4 \\ z e^{\pi i t/2} & |z| \geq 3/4 \end{cases}$$

Let $J'' = T \setminus \psi(-3/4, 3/4)$

Let $\lambda : T^n \longrightarrow [0,1]$ be continuous s.t. $\lambda|_{(J'')^n} = 1$,

$\lambda|_{T^n \setminus (J')^n} = 0$.

35 Define

$g'_{n+1}|_{(T^n\setminus J^n)\times T\times[-1,1]} \longrightarrow R^{n+1}\times R$ by

$g'_{n+1}(x,t,u) = (1\times\theta_{\lambda(x)})(\phi'_{n+1}(x,t),u))$

and define $g'_{n+1}|_{T^n\times(T-J)\times[-1,1]} \longrightarrow R^n \cancel{\times} R$

by $g'_{n+1}(x,t,u) = (f'_{n+1}\times 1)(x,(\psi x)\theta_{\lambda(x)}(\psi^{-1}(t),u))$

Can check that $g'_{n+1}|_{T^{n+1}\setminus(J')^{n+1}}$ agrees with $f'_{n+1}\times 1$.

Define $g'_{n+1}|_{(J')^{n+1}}$ to be the restriction of

$\sigma_n(1\times\phi)\sigma_n(g_n\times 1)\sigma_{n+1}(1\times T) : T^n\times T\times[-1,1]\to R^{n+2}$

where σ_j swaps jth & $(j+1)$th factors in $(n+2)$ fold product and $T:[-1,1]\to[-1,1]$ changes sign.

Supposes f_n, g_n constructed.

□

36

§ 4. Local contractibility

__Definition 4.1__ A space X is __locally contractible__ if for each point $x\in X$ and each nbhd U of x, $\exists$ nhd V of x and homotopy $H: V\times I\longrightarrow U$ such that $H_0 = 1$, $H_1(V) = x$.

□

Let X^I be set of paths in X ending at x.
Enough to find nbhd V and map

$\phi: V\longrightarrow X^I$ s.t. $\phi(y)$ is a path from y to x,

and $\phi(x)$ = constant path at x.

(Given open nbhd U of x, U^I is open set in X^I, so $\exists$ nbhd V' of x in X s.t. $\phi(V')\subset U^I$).

If M is a manifold, let $\mathcal{H}(M)$ be the space of homeomorphisms of M, together with the compact-open topology.

(37)

Definition 4.2 An <u>isotopy</u> of M is a path in $\mathcal{H}(M)$. Equivalently, an isotopy is a homeomorphism $H : M \times I \longrightarrow M \times I$ such that $p_2 H = p_2$. We say that H is an <u>isotopy from H_0 to H_1</u>, and H_0, H_1 are <u>isotopic</u>. $\square$

Theorem 4.3 $\mathcal{H}(M)$ is locally contractible. (Černavsky, Kirby).

<u>Proof</u>: $\mathcal{H}(\mathbb{R}^n)$ is a group, so it is enough to show that it is locally contractible at 1.

Choose embedding $i : 4B^n \longrightarrow T^n$ and choose immersion $f : T^n \setminus i(0) \longrightarrow \mathbb{R}^n$.

$T^n \setminus i(\mathrm{Int}\, B^n)$ is compact, $\exists\, \delta > 0$ s.t., for all $x \in T^n \setminus i(\mathrm{Int}\, B^n)$, $f|_{N_\delta(x)}$ is injective.

We may suppose $\delta < d(i(3B^n \setminus \mathrm{Int}\, 2B^n), i(4S^{n-1} \cup S^{n-1}))$

Since f is open $\varepsilon_x = d(f(x), \mathbb{R}^n \setminus N_\delta(f(x))) > 0$

$$\varepsilon = \inf\{\varepsilon_x \mid x \in T^n \setminus i(\mathrm{Int}\, B^n)\} > 0$$

If $x \in T^n \setminus i(\mathrm{Int}\, B^n)$ and $v \in \mathbb{R}^n$ are such that $d(f(x), v) < \varepsilon$, then $\exists$ unique $u \in N_\delta(x)$ such that $f(u) = v$.

(38)

Let $h \in \mathcal{H}(\mathbb{R}^n)$: suppose h so close to 1 that $d(h(f(x)), f(x)) < \varepsilon$ for all $x \in T^n \setminus i(\mathrm{Int}\, B^n)$.

For $x \in T^n \setminus i(\mathrm{Int}\, 2B^n)$, let $h'(x)$ be unique point in $N_\delta(x)$ such that $f h'(x) = h f(x)$, $h'(x) \in T^n \setminus i(\mathrm{Int}\, B^n)$

$$\begin{tikzcd}
\cancel{T^n \setminus i(\mathrm{Int}\,2B^n)} \\
\mathbb{R}^n \arrow[r, dashed, "h"] \arrow[d, "e"'] & \mathbb{R}^n \arrow[d, "e"] \\
T^n \arrow[r, dashed, "h"] & T^n \\
T^n \setminus i(\mathrm{Int}\,2B^n) \arrow[u, phantom, sloped, "\subset"] \arrow[r, "h'"] \arrow[d, "f"'] \arrow[ur, phantom, "\text{commutes on } T^n\setminus i(\mathrm{Int}\,3B)"] & T^n \setminus i(\mathrm{Int}\,B^n) \arrow[u, phantom, sloped, "\subset"] \arrow[d, dashed, "f"] \\
\mathbb{R}^n \arrow[r, dashed, "h"] \arrow[ur, phantom, "\text{commutes}"] & \mathbb{R}^n
\end{tikzcd}$$

Since f is an open immersion, h' is an open immersion. If $h'(x) = h'(y)$, then
$x, y \in N_\delta(h'(x)) \Rightarrow f(x) \neq f(y)$
$\Rightarrow h'(x) \neq h'(y)$ #

$\therefore$ h' is an embedding depending continuously on $h \in \mathcal{H}(\mathbb{R}^n)$

Consider $i^{-1} h' i : 3B^n \setminus \mathrm{Int}\, 2B^n \longrightarrow \mathrm{Int}\, 4B^n$.

By Cor 2.13 $\exists$ nbhd W of 1 in $\mathcal{E}(3B^n \setminus \mathrm{Int}\, 2B^n, \mathrm{Int}\, 4B^n)$ and continuous map $\phi : W \longrightarrow \mathcal{E}(3B^n, \mathrm{Int}\, 4B^n)$ s.t. $\phi(g)|_{3S^{n-1}} = g|_{3S^{n-1}}$.

Define $h'' : T^n \to T^n$; $x \longmapsto \begin{cases} h'(x) & x \notin i(3B^n) \\ i\phi(i^{-1}h'i)i^{-1} & x \in i(3B^n) \end{cases}$

Then $h'' : T^n \to T^n$ is a homeo, depending continuously on $h \in V$, where

$$V = \{h \in \mathcal{H}(\mathbb{R}^n) \mid h' \text{ is defined and } i^{-1} h' i^{-1} \in W\}$$

If V is sufficiently small, then $h \in V \Rightarrow h'' \underset{\text{homotopic only}}{\simeq} 1$

Let $e : \mathbb{R}^n \to T^n$ be the (universal) covering map.

By 3.3 $\exists$ homeo $\tilde{h} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $e\tilde{h} = h'' e$

If V is sufficiently small, there is a unique choice of $\tilde{h}$ such that $d(\tilde{h}(0), 0) < \frac{1}{2}$. Then $\hat{h}$ depends continuously on h.

(39)

By 3.3 $\tilde{h}$ ~~continuously~~ commutes with covering translations.

Let $I = [0,1]$: then every point of $\mathbb{R}^n$ can be moved into I^n by covering translations.

If $A = \sup_{x \in I^n} d(\tilde{h}(x), x) < \infty$, we have $d(\tilde{h}(x), x) \leq A$ for all x in $\mathbb{R}^n$. (i.e. $\tilde{h}$ is <u>bounded</u> homeo of $\mathbb{R}^n$)

$\updownarrow$

This means $d(x, \tilde{h}(x)) \leq A \quad \forall x \in \mathbb{R}^n$.

Suppose (w.l.o.g.) $e(0) \notin i(4B^n)$. Choose, once for all, $r > 0$ s.t. $f|_{e(rB^n)}$ is injective and $r < 1$ and $e(rB^n) \cap i(4B^n) = \emptyset$.

Define $\rho : \operatorname{Int} B^n \to \mathbb{R}^n$ by

$$\rho(x) = x \quad (x \in rB^n)$$
$$\rho(x) = \frac{r-1}{|x|-1} x \quad (x \notin rB^n)$$

ρ homeo $\operatorname{Int} B^n \to \mathbb{R}^n$, fixed on rB^n.

$\rho^{-1}\tilde{h}\rho$ is a homeomorphism from $\operatorname{Int} B^n \to \operatorname{Int} B^n$.

Suppose $|x|$ close to 1, (<1). Then

$$d(x, \rho^{-1}\tilde{h}\rho(x)) \leq \frac{2A(|x|-1)}{r-1} \text{ if } |x| \text{ sufficiently near } 1$$
$$\longrightarrow 0 \text{ as } |x| \to 1.$$

So $\rho^{-1}\tilde{h}\rho$ extends to a homeomorphism of B^n, fixed on ∂B^n.

Define isotopy R_t of B^n by

$$R_t(x) = \begin{cases} x & (|x| > t) \\ t\rho^{-1}\tilde{h}\rho(\frac{x}{t}) & (|x| < t) \end{cases}$$

The composite

$$\operatorname{Int} B^n \xrightarrow{\rho} \mathbb{R}^n \xrightarrow{e} T^n \longleftrightarrow T^n \setminus i(\operatorname{Int} 2B^n) \xrightarrow{f} \mathbb{R}^n$$

agrees with a homeomorphism $\sigma : \mathbb{R}^n \to \mathbb{R}^n$.

$\cancel{\sigma : \mathbb{R}^n \to \mathbb{R}^n}$.

(i.e. extend $fe|_{rB^n} \to \mathbb{R}^n$ to homeo $\sigma : \operatorname{Int} B^n \to \mathbb{R}^n$ e.g. by Schönflies theorem).

(40)

Choose s, $0 < s < r$. If V is small enough,

$$h \in V \Rightarrow \tilde{h}(sB^n) \subset \operatorname{Int} tB^n.$$

Define isotopy S_t of $\mathbb{R}^n$ by

$$S_t(x) = \sigma R_t \sigma^{-1}(x).$$

This depends continuously on h. $S_0 = 1$.

$$S_1|_{fe(sB^n)} = h|_{fe(sB^n)}$$

$$x \longrightarrow \underset{\in rB^n}{\sigma^{-1}x} \xrightarrow{\rho} \sigma^{-1}x \longrightarrow \underset{\in rB^n}{\tilde{h}\sigma^{-1}x} \longrightarrow \underset{= hx}{\sigma\tilde{h}\sigma^{-1}x}$$

W.l.o.g. $0 \in \operatorname{Int} fe(sB^n)$. $S_1^{-1}h$ is 1 on a nbhd of 0.

Define $F_t : \mathbb{R}^n \to \mathbb{R}^n$ by

$$F_t(x) = \begin{cases} t^{-1}S^{-1}h(tx) & (t \neq 0) \\ x & (t = 0) \end{cases}$$

Define $H_t = S_t F_t$ (i.e. $H_t(x) = S_t(F_t(x))$).

This is isotopy from 1 to h.

H_t depends continuously on $h \in V$, and

$$h = 1 \Rightarrow H_t = 1,$$

So $\mathcal{H}(\mathbb{R}^n)$ <u>is</u> locally contractible

$\square$

(41) What about $\mathcal{H}(M)$ for (say) M compact?

(Use handle decomposition)

Let $\mathcal{E}(k\text{-handle})$ be space of embeddings of $B^k \times B^n \to B^k \times \mathbb{R}^n$ leaving $(\partial B^k) \times B^n$ fixed.

<u>Theorem 4.4</u> There is a neighbourhood V of 1 in $\mathcal{E}(k\text{-handle})$ and a homotopy $H : V \times I \to \mathcal{E}(k\text{-handle})$ s.t.

i) $H_t(1) = 1 \quad (\forall t)$

ii) $H_0(h) = h \quad (\forall h \in V)$

iii) $H_1(h)|_{B^k \times \frac{1}{2}B^n} = 1$

iv) $H_t(h)|_{\partial\{B^k \times B^n\}} = h|_{\partial\{B^k \times B^n\}} \quad \forall t, h$.

<u>Proof</u>: Let $i : 4B^n \to T^n$ be fixed embedding,

$f : T^n \setminus \{0\} \to \cancel{\mathbb{R}}\ \operatorname{Int} \frac{1}{2}B^n$ fixed immersion.

Choose $r > 0$ s.t. $r < 1$, $f|_{e(rB^n)}$ injective $e(rB^n) \cap i(4B^n) = \emptyset$

Modify f so that $f(e(\operatorname{int} rB^n)) \supset \frac{1}{2}B^n$.

$$\begin{tikzcd}
B^k \times \mathbb{R}^n \arrow[r, "\tilde{g}"] \arrow[d, "1 \times e"] & B^k \times \mathbb{R}^n \arrow[d, "1 \times e"] \\
B^k \times T^n \arrow[r, "g''"] & B^k \times T^n \\
B^k \times (T^n \setminus i(\operatorname{int} 2B^n)) \arrow[u] \arrow[r, "g'"] \arrow[d] \arrow[ur, phantom, "\copyright \text{ on } B^k \times (T^n \setminus i(\operatorname{int} 3B))"] & B^k \times (T^n \setminus i(\operatorname{int} B^n)) \arrow[u] \arrow[d] \\
B^k \times B^n \arrow[r, "g \in V"] & B^k \times \mathbb{R}^n
\end{tikzcd}$$

(42) Let $h \in \mathcal{E}(k\text{-handle})$ be close to 1.

First preliminary is ~~hom~~otopy G from h to $g \in \mathcal{E}(k\text{-handle})$ such that $g|_{\overline{B^k \setminus \frac{1}{2}B^k} \times \frac{3}{4}B^n} = 1$.

$$G_t(x,y) = \begin{cases} (x,y) & (|x| \geq 1 - t/2) \\ \left((1-\tfrac{t}{2})\, h_1((1-\tfrac{t}{2})^{-1}x, y),\ h_2(x,y)\right) & |x| \leq 1 - \tfrac{t}{2} \end{cases}$$ where $h(x,y) = (h_1(x,y), \ldots$

$G_0 = h$, $G_1 = g$ is an embedding fixed on $\overline{B^k \setminus \frac{1}{2}B^k} \times \frac{3}{4}B^n$. G depends continuously on h

<u>THIS DEFN. HAS TO BE</u> <u>MODIFIED</u> <u>TO HAVE $\frac{3}{4}$</u>

AND ALSO

and

$$G_t|_{B^k \times \partial B^n} = h|_{B^k \times \partial B^n}$$

As in 4.3 construct embedding

$$g' : B^k \times (T^n \setminus i(\operatorname{int} 2B^n)) \to B^k \times (T^n \setminus i(\operatorname{int} B^n))$$

s.t. $(1 \times f)\, g' = g(1 \times f)$,

and $g'|_{(\overline{B^k \setminus \frac{1}{2}B^k}) \times (T^n \setminus -)} = 1$.

Put $g'|_{\overline{B^k \setminus \frac{1}{2}B^k} \times T^n} = 1$. This extends the g' already defined.

Use 2.13 to extend

$$g'|_{[\frac{3}{4}B^k \times i(3B^n) \setminus \operatorname{int}(\frac{1}{2}B^k \times i(2B^n))]}$$

to embedding

$$g'' : \tfrac{3}{4}B^k \times i(3B^n) \to B^k \times i(4B^n)$$

43 s.t. $g'' = g'$ on $\partial(\frac{3}{4}B^k \times i(3B^n))$.

Let $\tilde{g} : B^k \times \mathbb{R}^n \longrightarrow B^k \times \mathbb{R}^n$ be such that

$(1 \times e)\tilde{g} = g''(1 \times e)$ and $\tilde{g}|_{\partial B^k \times B^n} = 1$.

$\tilde{g}$ is bounded, i.e.

$$d(x, \tilde{g}(x)) \leqslant A \qquad (x \in B^k \times \mathbb{R}^n)$$

Extend $\tilde{g}$ to a homeo of $\mathbb{R}^k \times \mathbb{R}^n$ by

$\tilde{g}|_{(\mathbb{R}^k \setminus B^k) \times \mathbb{R}^n} = 1$.

$\rho : \operatorname{int}(2B^k \times 2B^n) \longrightarrow \mathbb{R}^k \times \mathbb{R}^n$, homeomorphism fixing $B^k \times B^n$,

$$\rho(x,y) = \begin{cases} (x,y) & ((x,y) \in B^k \times B^n) \\ (2 - \max(|x|,|y|))^{-1}(x,y) & ((x,y) \notin B^k \times B^n) \end{cases}$$

Then $\rho^{-1}\tilde{g}\rho : \operatorname{int}(2B^k \times 2B^n) \circlearrowleft$ extends to a homeomorphism of $2B^k \times 2B^n$, fixed on $\partial(2B^k \times 2B^n)$.

In fact, $\rho^{-1}\tilde{g}\rho$ fixes $(2B^k \setminus \operatorname{int} B^k) \times 2B^n$.

Thus $\rho^{-1}\tilde{g}\rho$ defines homeo of $B^k \times 2B^n$ fixed on $\partial(B^k \times 2B^n)$. Define isotopy R_t of $B^k \times 2B^n$ by

$$R_t(x,y) = \begin{cases} (x,y) & \max(|x|, \frac{1}{2}|y|) \geqslant t \\ t\rho^{-1}\tilde{g}\rho(t^{-1}(x,y)) & \max(|x|, \frac{1}{2}|y|) \leqslant t \end{cases}$$

Let $\sigma : B^k \times 2B^n \longrightarrow B^k \times \operatorname{int} B^n$ be an embedding with $\sigma|_{B^k \times rB^n} = fe$

44 Now define isotopy S_t of $B^k \times B^n$ by

$$S_t(x) = \begin{cases} \sigma R_t \sigma^{-1} & (x \in \operatorname{im}\sigma) \\ x & (x \notin \operatorname{im}\sigma) \end{cases}$$

Then $S_0 = 1$, and S_t fixes $\partial(B^k \times B^n)$.

Suppose V is so small that

$h \in V \Rightarrow \tilde{g}$ is defined and also $g(\frac{1}{2}B^n) \subset fe(\operatorname{int} rB^n)$.

Then $S_1|_{B^k \times \frac{1}{2}B^n} = g$.

Define

$$H_t : B^k \times B^n \to B^k \times \mathbb{R}^n;\ x \longmapsto \begin{cases} G_{2t}(x) & (t \leq \frac{1}{2}) \\ gS^{-1}_{2t-1}(x) & (\frac{1}{2} \leq t \leq 1) \end{cases}$$

This does what is required.

□

<u>Lemma 4.5</u> If $C \subset \mathbb{R}^n$ is compact and $\epsilon > 0$, then C lies in the interior of a handlebody with handles of diameter $< \epsilon$. Explicitly, $\exists$ finitely many embeddings $h_i : B^{k_i} \times B^{n-k_i} \longrightarrow \mathbb{R}^n$ $(i = 1, 2, \ldots, \ell)$ such that, if $W_j = \bigcup_{i \leq j} h_i(B^{k_i} \times \frac{1}{2}B^{n-k_i})$ then

i) $h_i(B^{k_i} \times B^{n-k_i}) \cap W_{i-1} = h_i(\partial B^{k_i} \times B^{n-k_i})$

ii) W_ℓ is a neighbourhood of C

iii) $h_i(B^{k_i} \times B^{n-k_i})$ has diameter $< \epsilon$, and $h_i(B^{k_i} \times B^{n-k_i}) \subset N_\epsilon(C$

(45) **Proof**: Cover C by a lattice of cubes of side $\frac{1}{2}\epsilon$. Since C is compact, C only needs a finite no. of these. Let $\gamma_1, \dots, \gamma_l$ be all the faces of all the cubes ~~of the~~ meeting C.

Let $k_i = \dim \gamma_i$ and order γ_i so that $k_0 \le k_1 \le \dots \le k_l$.

Define metric on $\mathbb{R}^n$ by $d((x_1,\dots,x_n),(y_1\dots y_n)) = \max_{1\le i\le n}|x_i - y_i|$

Let $H_i = N_{\epsilon 2^{-i-3}}(\gamma_i) \setminus \bigcup_{j<i} N_{\epsilon 2^{-j-4}}(\gamma_j)$

and $\frac{1}{2}H_i = H_i \cap N_{\epsilon 2^{-i-4}}(\gamma_i)$

Then $H_i \cap \gamma_i \cong \gamma_i$ (radial projection) $\cong B^{k_i}$

and clearly $H_i \cong (H_i \cap \gamma_i) \times B^{n-k_i}$

$\exists$ homeo $h_i : B^{k_i} \times B^{n-k_i} \to H_i$ carrying $B^{k_i} \times \frac{1}{2}B^{n-k_i}$ onto $\frac{1}{2}H_i$ and $(\partial B^{k_i}) \times B^{n-k_i}$ onto $H_i \cap \bigcup_{j<i}(\frac{1}{2}H_j)$.

Then $h_1, h_2 \dots h_l$ do what is required. $\square$

Addendum 4.6 If $D \subset C$ is compact, then we can select $h_{i_1}, \dots h_{i_m}$ so that i) is still satisfied, ii) and iii) satisfied by $h_{i_1} - h_{i_m}$ w.r.t. D instead of C, i.e. $\bigcup h_{i_r}(B^{k_{i_r}} \times \frac{1}{2}B^{n-k_{i_r}})$ is a nbhd of D, $h_{i_r}(B^{k_{i_r}} \times B^{n-k_{i_r}})$ has diam $< \epsilon$ and is $\subset N_\epsilon(D)$

Proof: Select h_i iff γ_i is a face of a cube which meets D. $\square$

(46) **Theorem 4.7** (Kirby - Edwards) Let C, D be compact in $\mathbb{R}^n$ and U, V be nbhds of C, D. Let $\mathcal{E}$ be space of embeddings of U in $\mathbb{R}^n$ which restrict to 1 on V. There is a neighbourhood N of 1 in $\mathcal{E}$ and a homotopy $H : N \times I \to \mathcal{H}(\mathbb{R}^n)$ s.t.

i) $H_t(1) = 1 \quad \forall t$

ii) $H_0(g) = 1 \quad \forall g \in N$

iii) $H_1(g)|_C = g|_C$

iv) $H_t(g)|_{D \cup (\mathbb{R}^n \setminus U)} = 1 \quad \forall t, g$

Proof: Let $\epsilon = \min(d(C, \mathbb{R}^n\setminus U), d(D, \mathbb{R}^n \setminus V))$.

Cover ~~C~~ $C \cup D$ by a handlebody in $U \cup V$, handles of diameter $< \epsilon$. Let $h_1 - - h_l$ be the handles, W_i as in 4.5. Select $h_{i_1} \dots h_{i_m}$ to form sub-handlebody covering D, contained in V.

Let $X = \bigcup_r h_{i_r}(B^{k_{i_r}} \times B^{n-k_{i_r}})$ $\quad (W_i = \bigcup_{j<i} h_j(B^{k_j} \times \frac{1}{2}B^{n-k_j})$

Suppose inductively that we have constructed nbhd. N_{i-1} of 1 in $\mathcal{E}$, and homotopy $H^{(i-1)} : N_{i-1} \times I \to \mathcal{H}(\mathbb{R}^n)$ such that i) and ii) are satisfied;

$H_1^{(i-1)}(g)|_{W_{i-1}} = g$, $H_t^{(i-1)}(1) = 1$, $H_0^{(i-1)}(g) = 1$,

$H_r^{(i-1)}(g)|_{X \cup (\mathbb{R}^n \setminus U)} = 1$.

If $h_i(B^{k_i}\times B^{n-k_i}) \subset X$, put $N_i = N_{i-1}$ and $H_i = H_{i-1}$. (This is consistent because if $h_i(B^{k_i}\times B^{n-k_i}) \cap h_j(B^{k_j}\times B^{n-k_j}) \neq \emptyset$ for $j<i$ then $h_j(B^{k_j}\times B^{n-k_j}) \subset X$).

Now suppose $h_i(B^{k_i}\times B^{n-k_i}) \not\subset X$.
Choose N_i small so that $g^{-1}H_t^{(i-1)}(g)\, h_i(B^{k_i}\times \frac{3}{4}B^{n-k_i}) \subset h_i(B^{k_i}\times \mathrm{int}\, B^{n-k_i})$

Let $f = h_i^{-1} g_i^{-1} H_1^{(i-1)}(g)\, h_i : B^{k_i}\times\frac{3}{4}B^{n-k_i} \to B^{k_i}\times \mathrm{int}\, B^{n-k_i}$

Then f fixes $(\partial B^{k_i})\times \frac{3}{4}B^{n-k_i}$. Theorem 4.4 gives a continuously varying isotopy $H'_t(g)$ s.t.

a) $H'_t(1) = 1$

b) $H'_0(g) = f$

c) $H'_1(g)|_{B^{k_i}\times\frac{1}{2}B^{n-k_i}} = \cancel{f|_{B^{k_i}\times\frac{1}{2}B^{n-k_i}}}\ 1$

d) $H'_t(g)|\partial(B^{k_i}\times\frac{3}{4}B^{n-k_i}) = 1$.

$\cancel{H^{i-1}: N_{i-1}\times I \to \mathcal{H}(\mathbb{R}^n)}$ s.t.

i) $\cancel{H_t^{(i-1)}(1) = 1}$

ii) $\cancel{H_0(g) = 1}$

iii) $\cancel{H^{(i-1)}|_{W_{i-1}} = g}$

iv) $\cancel{H_t^{(i-1)}|_{X\cup\mathbb{R}^n\setminus U} = \cancel{g}\ 1}$

Define $H_t^{(i)}(g)(x) = (H_t^{(i-1)}(g))\, h_i f^{-1}(H'_t(g))\, h_i^{-1}(x)$ $\quad (x\in h_i(B^{k_i}\times\frac{3}{4}B^{n-k_i}))$

and $\quad H_t^{(i)}(g)(x) = H_t^{(i-1)}(g)(x) \quad (x\notin$ ——— $)$

$W_i = W_{i-1}\cup h_i(B^{k_i}\times\frac{1}{2}B^{n-k_i})$

$h_i(B^{k_i}\times\frac{3}{4}B^{n-k_i})\cap X \subset h_i(\partial(B^{k_i}\times\frac{3}{4}B^{n-k_i}))$

Completes induction.

$H = H^\ell$, $N = N^\ell$ does what is required. □

Theorem 4.8 If M is a compact manifold then $\mathcal{H}(M)$ is locally contractible.

Proof: First suppose M is closed ($\partial M = \emptyset$). Cover M by finitely many embeddings $f_i : \mathbb{R}^n \to M$ $(i = 1, 2, \dots, \ell)$. In fact, assume $M = \bigcup f_i(B^n)$.

Let $h: M\to M$ be homeo near 1.

Define inductively isotopy $H^{(i)}(h)$ of M such that

i) $H_t^{(i)}(h)$ depends continuously on h

ii) $H_t^{(i)}(1) = 1$ iii) $H_0^{(i)}(h) = 1$ iv) $H_1^{(i)}(h)$ agrees with h on $\bigcup_{j\leq i} f_j((1+2$

(49)

Suppose $H_t^{(i-1)}$ defined. Let $C = (1+2^{-i})B^n$,

$U = (1+2^{-(i-1)})B^n$, and let $D = f_i^{-1}\left(\bigcup_{j<i} f_j\left((1+2^{-i})B^n\right)\right) \cap 4B^n$

$V = f_i^{-1}\left(\bigcup_{j<i} f_j\left((1+2^{-(i-1)})B^n\right)\right)$.

Suppose h is so near 1 that $h^{-1}H_t^{(i-1)}(h)\, f_i(U) \subset f_i(\mathbb{R}^n)$

Apply Thm 4.7 to $g = f_i^{-1} h^{-1} H_1^{(i-1)}(h) f_i : U \longrightarrow \mathbb{R}^n$

If h sufficiently near 1, we get continuously varying isotopy $H'(h)$ of $\mathbb{R}^n$ s.t.

a) $H_t'(1) = 1 \quad \forall t$

b) $H_0'(h) = 1 \quad \forall h$

c) $H_1'(h)|_C = \overset{h}{g}|_C$

d) $H_t'(h)|_{D \cup (\mathbb{R}^n \setminus U)} = 1$ all t, h.

Define $H^{(i)} = H^{(i)}(h)$ by ~~$H_t^{(i)}(x) = H^{(i-1)}$~~,

$H_t^{(i)}(x) = H_t^{(i-1)} f_i\,(H_t'(h))^{-1} f_i^{-1}(x)$ (if $x \in f_i(\mathbb{R})$)

$= H_t^{(i-1)}(x)$ (if $x \notin$ —)

Then $H^{(i)}$ satisfies i) – iii), and completes induction.

(50) Now suppose $\partial M \neq \emptyset$. Let

$$\gamma : \partial M \times I \longrightarrow M$$

be a collar of ∂M in M. $\mathcal{H}(\partial M)$ is locally contractible. If $h \in \mathcal{H}(M)$ near 1 then we have isotopy $H_t(h)$ of ∂M with $H_0(h) = 1$, $H_1(1) = h|_{\partial M}$.

Define isotopy $\bar{H}$ of M by

$$\bar{H}_t(\gamma(x,u)) = \gamma(H_{t(1-u)}(x), u)$$

$$x \in \partial M,\ u \in I.$$

$$\bar{H}_t(y) = y \text{ if } y \notin \gamma(\partial M \times I).$$

Then $\bar{H}_t$ is an isotopy of M from 1 to $\bar{H}_1$, where $\bar{H}_1$ agrees with h on ∂M.

$\exists$ isotopy $G_t : M \to M$ from $\bar{H}_1$ to G_1, where G_1 agrees with h on $\gamma(\partial M \times [0, \frac{1}{2}])$.

Now argument goes as for closed manifolds

$\square$

<u>Exercise</u>: If M is compact then $\mathcal{H}(\operatorname{Int} M)$ is locally contractible.

$\square$

Theorem 4.9 (Isotopy extension) Let M, N be n-manifolds with M compact, $\partial N = \emptyset$ and $M \subset N$. Suppose given a path $H : I \longrightarrow \mathcal{E}(M,N)$ (and $H_0 : M \hookrightarrow N$). If U is a neighbourhood of ∂M in M, then $\exists$ isotopy $\bar{H} : I \longrightarrow \mathcal{H}(N)$ such that $\bar{H}_0 = 1$ and $\bar{H}_t|_{M\setminus U} = H|_{M\setminus U}$.

Proof: First use method of 4.8 to generalize 4.7 to ~~closed~~ deal with compact $C, D \subset N$ (i.e. replace $\mathbb{R}^n$ by N).

Let $f \in \mathcal{E}(M,N)$. Then $f(M) \subset N$ is a neighbourhood of $f(M\setminus U)$ (assume U open), and $\exists$ open nbhd V_f of 1 in $\mathcal{E}(f(M), N)$ and homotopy $F^{(f)} : V_f \times I \longrightarrow \mathcal{H}(N)$ s.t. $F^{(f)}_1(g)|_{M\setminus U} = g|_{M\setminus U}$ for $g \in V_f$.

Let $W_f = \{ gf \mid g \in V_f \}$. Then W_f is an open neighbourhood of f in $\mathcal{E}(M,N)$. Now $\{W_f\}_{f \in \mathcal{E}(M,N)}$ is an open cover of $\mathcal{E}(M,N)$.
$\exists$ dissection $0 = t_0 < t_1 < \dots < t_L = 1$ of I s.t.

$H[t_{i-1}, t_i] \subset$ some W_{f_i} $\quad f_i \in \mathcal{E}(M,N)$.

Define $\bar{H}_t$ for $t_{i-1} \leq t \leq t_i$ by

$$\bar{H}_t = F_1^{(f_i)}(H_t \circ f_i^{-1}) \left(F_1^{(f_i)}(H_{t_{i-1}} \circ f_i^{-1})\right)^{-1} \bar{H}_{t_{i-1}}.$$

Then $\bar{H}_t = H_t$ on $M \setminus U$.

$\square$

Addendum 4.10 $\bar{H}_t$ can be chosen to be the identity outside some compact set. $\square$

(because 4.7 also produces isotopies of compact support).

Corollary 4.11 Let $f : B^n \hookrightarrow \operatorname{Int} 2B^n$ be isotopic to 1. Then $2B^n \setminus f(\operatorname{Int} \frac{1}{2}B^n) \cong 2B^n \setminus \operatorname{Int} \frac{1}{2}B^n$.

Proof: Let H_t be an isotopy from 1 to f.

By 4.10 $\exists$ isotopy $\bar{H}_t$ of $\operatorname{Int} 2B^n$, fixed outside λB^n for some $\lambda < 2$, such that $\bar{H}_1 = f$ on $\frac{1}{2}B^n$.

$\therefore$ $\tilde{H}_1$ defines homeomorphism

$2B^n \setminus \operatorname{Int} \frac{1}{2}B^n \longrightarrow 2B^n \setminus f(\operatorname{Int} \frac{1}{2}B^n)$

$\square$

§5 Triangulation theorems

<u>Defn 5.1</u> An <u>r-simplex</u> in $\mathbb{R}^n$ is the convex hull of $(r+1)$ linearly independent points.

Let $K \subset \mathbb{R}^n$ be compact. An embedding $f : K \to \mathbb{R}^n$ is <u>PL</u> if K is a finite union of simplexes, each mapped linearly by f.

If M is an n-manifold ~~dt~~, a <u>PL structure</u> on M is a family $\mathcal{F}$ of embeddings $f : \Delta^n \to M$ s.t.

i) Every point of M has a neighbourhood of from $f(\Delta^n)$ $(f \in \mathcal{F})$

ii) If $f, g \in \mathcal{F}$ then $g^{-1}f \,|\, f^{-1}g(\Delta^n) \to \mathbb{R}^n$ is PL

iii) $\mathcal{F}$ is maximal w.r.t. i), ii).

If M, N have PL structures $\mathcal{F}, \mathcal{G}$ an embedding $h : M \to N$ is <u>PL</u> if $f \in \mathcal{F} \Rightarrow hf \in \mathcal{G}$.

<u>Examples</u> i) The composite of 2 PL embeddings is PL, ~~and distributive over comp~~ i.e. PL $\cong$ is equivalence reln.

ii) A PL structure $\mathcal{F}$ on M defines a PL structure $\partial\mathcal{F}$ on ∂M

iii) A compact n-manifold ~~is~~ has PL structure $\iff$ it has a triangulation with

$$\text{link}(\text{vertex}) \underset{PL}{\cong} \partial\Delta^n.$$

We need 3 deep theorems from PL topology

<u>Proposition 5.2</u>

(A) Suppose M is a closed PL manifold which is ~~closed~~ hty. equivalent to S^n. If $n \geq 5$, then M is PL homeomorphic to $S^n = \partial\Delta^{n+1}$.

(B) Call a non-compact manifold W <u>simply-connected at ∞</u> if for every compact set $C \subset W$, $\exists$ compact set $D \subset W$ s.t. any two loops in $W \setminus D$ are homotopic in $W \setminus C$.

(example: $\mathbb{R}^n$ is simply connected at ∞ iff $n \geq 3$)

Suppose W^n is an open PL manifold which is simply connected at infinity. If $n \geq 6$ then W is PL homeomorphic to Int V where V is some compact PL manifold. (Browder, Levine, Livesay)

(C) Let M be a closed PL manifold which is homotopy equivalent to T^n. Then some finite covering of M is PL homeomorphic to $T^n = (\partial\Delta^2)^n$ (Proof in Wall's book). □

Theorem 5.3 (Annulus conjecture) If $h: B^n \to \mathrm{Int}\, B^n$ is an embedding and $n \geq 6$, then $B^n \setminus h(\mathrm{Int}\, \frac{1}{2}B^n) \cong B^n \setminus \mathrm{Int}\, \frac{1}{2}B^n$

Proof: Let $a \in T^n$ and let $f: T^n \setminus a \to \mathrm{Int}\, B^n$ be a PL immersion s.t. $f(T^n \setminus a) \subset \frac{1}{2}B^n$.
Let $h: B^n \to \mathrm{Int}\, B^n$ be top homeomorphism: we shall find PL structure $\mathcal{F}'$ on $T^n - a$ s.t. hf is PL w.r.t. $\mathcal{F}'$.
Let $\mathcal{F}_0 = \{\phi: \Delta^n \to T^n \setminus a \mid (hf)\phi \text{ is PL embedding}\}$
Since hf is an open immersion, $\{\phi(\mathrm{Int}\,\Delta^n) \mid \phi \in \mathcal{F}\}$ covers $T^n \setminus a$ $\quad \phi, \psi \in \mathcal{F} \Rightarrow \psi^{-1}\phi: \phi^{-1}\psi(\Delta^n) \to \mathbb{R}^n$ is PL.
Extend $\mathcal{F}_0$ to PL structure $\mathcal{F}'$ on $T^n \setminus a$. Let $(T^n - a, \mathcal{F}')$

For $n \geq 3$ $(T^n - a)' \simeq (T^n - a)$ so $(T^n - a)'$ is simply-connected at ∞. Since $n \geq 6$, by 5.2(B) $\exists$ compact PL manifold W and PL homeo $g: (T^n - a)' \to \mathrm{Int}\, W$.
$\exists$ PL collar $\gamma: \partial W \times I \to W$ $\quad \epsilon > 0$ small. Let A be a nbhd of a in T^n which is homeo to B^n, and so small that

$$g^{-1}\gamma(\partial W \times I) \supset \underbrace{A - a}_{\simeq S^{n-1}} \supset g^{-1}\gamma(\partial W \times \epsilon)$$

ht equivalent

It follows that $\partial W \simeq S^{n-1}$, so by 5.2(A) since $n \geq 6$, ∂W is PL homeo to S^{n-1}.

By Schönflies theorem, $a \cup g^{-1}\gamma(\partial W \times (0, \epsilon]) \cong B^n$

Extend $\mathcal{F}'|_{T^n - (a \cup g^{-1}\gamma(\partial W \times (0,\epsilon))}$ to PL structure $\mathcal{F}''$ on
[For $\mathcal{F}'$ induces PL structure on $\partial(a \cup g^{-1}\gamma(\partial W \times (0, \epsilon]))$; extend 'conewise' to PL structure on $a \cup g^{-1}\gamma(\partial W \times (0,\epsilon])$]
By 5.2(C), $\exists$ finite covering of $(T^n)''$ which is PL homeo to T^n. Let $\varepsilon'': T^n \to (T^n)''$ be a finite cover
Let $\varepsilon: T^n \to T^n$ be corresponding cover of T^n.
By theory of covering spaces $\exists$ homeo $\bar{h}: T^n \to T^n$ (not P
such that $\varepsilon = \varepsilon''\bar{h}$ ($\bar{h}$ homotopic to 1). Now let $\tilde{h}: \mathbb{R}^n \to \mathbb{R}^n$ be a homeo s.t. $e\tilde{h} = \bar{h}e$. ~~No~~ Then $d(x, \tilde{h}(x))$ is bounded uniformly for $x \in \mathbb{R}^n$.
Let $\rho: \mathrm{Int}\, B^n \to \mathbb{R}^n$ be a PL 'radial' homeomorphism (avoiding "standard mistake")
Now $\eta = \rho^{-1}\tilde{h}\rho: \mathrm{Int}\, B^n \to \mathrm{Int}\, B^n$ extends to a homeo of B^n fixing ∂B^n.
Let U be a non-empty open set in $\mathrm{Int}\, B^n$ such that $\varepsilon e\rho(U) \cap A = \emptyset$ and $\sigma = f\varepsilon e\rho|_U$ maps U injectively into $\frac{1}{2}B^n$.
Let $\sigma'' = hf\varepsilon'' e\rho|_{\eta(U)} \longrightarrow \mathrm{Int}\, B^n$
Then σ, σ'' are PL embeddings and $\sigma''\eta = h\sigma$
The PL annulus conjecture is true (proof by regular neighbourhood theory).
$\exists$ n-simplex $\Delta \subset U$ s.t. $\eta(\Delta) \subset$ some n-simplex $\Delta'' \subset \eta(U)$

(57)

$\therefore$ By PL annulus theorem,

$$\overline{\tfrac{1}{2}B^n \setminus \sigma(\Delta)} \cong \text{std. annulus} \cong \overline{B^n \setminus \tfrac{1}{2}B^n}$$

$$B^n - h(\operatorname{int} \tfrac{1}{2}B^n) \cong B^n - h\sigma(\operatorname{int} \Delta)$$

(obtained by glueing standard annulus $h(\frac{1}{2}B^n) - h\sigma(\operatorname{int}\Delta)$ onto $B^n - h(\operatorname{int}\frac{1}{2}B^n)$)

$$= B^n - \sigma'\eta(\operatorname{int}\Delta)$$
$$\cong \sigma''(\Delta'') - \sigma''\eta(\operatorname{int}\Delta)$$
$$\cong \Delta'' - \eta(\operatorname{int}\Delta) \cong B^n - \eta(\operatorname{int}\Delta)$$
$$\cong B^n - \operatorname{int}\Delta \cong B^n - \operatorname{int}\tfrac{1}{2}B^n$$

$\square$

The proof depends only on knowing that given embeddings $f, g : B^n \longrightarrow T^n$ $\exists h : T^n \longrightarrow T^n$ carrying $f(\frac{1}{2}B^n)$ onto $g(\frac{1}{2}B^n)$. If we could do this purely geometrically (i.e. without PL theory) for all dimensions, we would have then proved the annulus conjecture in all dimensions

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(58)

New notation:

W = any manifold

$\rfloor$ = subset $(\partial W \times I) \cup (W \times 1)$ of $W \times I$

Theorem 5.4 Let M be a PL manifold and let $h : I \times B^k \times \mathbb{R}^n \longrightarrow M$ be a homeomorphism which is PL on a nbhd of $\rfloor$. If $k+n \geqslant 6$ then $\exists$ isotopy

$H_t : I \times B^k \times \mathbb{R}^n \longrightarrow M$ s.t.

1) $H_0 = h$
2) H_1 is PL on $I \times B^k \times B^n$
3) $H_t = h$ on $\rfloor$ and outside $I \times B^k \times 2B^n$.

Proof: Let $a \in T^n$ and let $f : T^n - a \longrightarrow \mathbb{R}^n$ be a PL immersion. As in 5.3, let $\mathcal{F}'$ be a PL structure on $I \times B^k \times (T^n - a)$ s.t.

$$h(1 \times f) : (I \times B^k \times (T^n - a))' \longrightarrow M \text{ is PL}$$

Then $\mathcal{F}'$ agrees with $\mathcal{F}$ near $\rfloor$.

Let A be a ball nbhd of a in T^n

First extend $\mathcal{F}'$ over a nbhd. of ~~a in T^n~~ ~~A~~ $\rfloor$ in $I \times B^k \times T^n$ (using std. structure).

As in 5.3 extend $\mathcal{F}'$ over ~~$B\times B$~~ $0\times B^k\times T^n$, obtaining structure $\mathcal{F}''$. Collaring argument $\Rightarrow$ can extend $\mathcal{F}''\cup\mathcal{F}|_{I\times B^k\times(T^n-A)}$ over nbhd of $0\times B^k\times T^n$ in $I\times B^k\times T^n$.

As in 5.3 extend to PL structure over $I\times B^k\times T^n$ agreeing with standard structure near] and with $\mathcal{F}'$ on $I\times B^k\times(T^n-A)$.

We can take $\mathcal{F}''$ to be the standard structure near $1\times B^k\times T^n$. Now $\mathcal{F}''$ is defined near $\partial(I\times B^k\times A)$; we extend over $I\times B^k\times A$ as in 5.3, obtaining a PL manifold $(I\times B^k\times T^n)''$.

The inclusion $(I\times B^k\times(T^n-a))'\hookrightarrow(I\times B^k\times T^n)''$ is PL except on $I\times B^k\times A$, and the identity map

$$I\times B^k\times T^n\longrightarrow(I\times B^k\times T^n)''$$

is PL near].

Now we need another result from PL topology:

Proposition 5.5 Let W, V_1, V_2 be compact PL manifolds with $\partial W = V_1\cup V_2$, and $V_1\cap V_2=\partial V_1=\partial V_2$. Suppose the inclusions $V_i\longrightarrow W$ are hty. equivalences $(i=1,2)$. If $\pi_1(W)$ is free abelian, and $\dim W\geq 6$, then W is PL homeomorphic to $V_1\times I$.

$\square$

Apply this result with $W=(I\times B^k\times T^n)''$, $V_1=$] and $V_2=(0\times B^k\times T^n)''$. We obtain a PL homeomorphism $(I\times B^k\times T^n)''\longrightarrow\,]\times I$. Since $]\times I\cong I\times B^k\times T^n$ by a PL homeomorphism taking $(x,0)$ to x, we can find a PL homeomorphism $g: I\times B^k\times T^n\longrightarrow(I\times B^k\times T^n)''$ which is the identity near].

Let $\tilde h: I\times B^k\times\mathbb{R}^n\longrightarrow I\times B^k\times\mathbb{R}^n$ be such that $e\tilde h = g^{-1}e$ and $\tilde h = 1$ on]. Then $\tilde h$ is a bounded homeomorphism. Extend $\tilde h$ over $[0,\infty)\times\mathbb{R}^k\times\mathbb{R}^n$ by putting $\tilde h=1$ outside $I\times B^k\times\mathbb{R}^n$.

Extend further over $\mathbb{R}\times\mathbb{R}^k\times\mathbb{R}^n$ by putting $\tilde h(t,x,y)=(t, p_2\tilde h(0,x,y), p_3\tilde h(0,x,y))$ for $t\leq 0$.

Note that $d(x,\tilde h(x))$ remains bounded for $x\in\mathbb{R}\times\mathbb{R}^k\times\mathbb{R}^n$.

(61) Suppose $0<r<1$, $e(rB^n)\cap A=\emptyset$ and $fe|_{rB^n}$ is injective. We may also suppose $fe(rB^n)\supset sB^n$ for some $s>0$. There is a PL 'radial' homeomorphism $\rho: (-1,2)\times \mathrm{Int}(2B^k\times B^n)\longrightarrow \mathbb{R}\times\mathbb{R}^k\times\mathbb{R}^n$, fixed near $I\times B^k\times rB^n$. Then $\rho\hat{h}\rho^{-1}$ extends to a homeomorphism of $[1,2]\times 2B^k\times B^n$ fixing the boundary.

Let $\eta=\rho\hat{h}\rho^{-1}| \; I\times B^k\times B^n\longrightarrow I\times B^k\times B^n$.

Note that $I\times B^k\times B^n\times I$ is the join of $(\frac{1}{2},0,0,\frac{1}{2})$ to $(J\times I)\cup(I\times B^k\times B^n\times\partial I)$. Define PL homeomorphism R of $I\times B^k\times B^n\times I$ by

$$R(\tfrac{1}{2},0,0,\tfrac{1}{2})=(\tfrac{1}{2},0,0,\tfrac{1}{2})$$

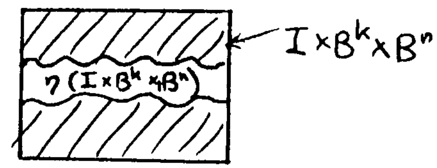
$R|(J\times I)\cup(I\times B^k\times B^n\times 1)=1$, $R|_{I\times B^k\times B^n\times 0}=\eta$, and extending conewise. Then R defines a PL isotopy R_t of $I\times B^k\times B^n$, fixed near J, with $R_0=\eta$ and $R_1=1$.

Let $\sigma: I\times B^k\times B^n\longrightarrow I\times B^k\times\mathbb{R}^n$ be a PL embedding which agrees with $1\times fe$ near $I\times B^k\times rB^n$. Then $h\sigma\eta^{-1}$ agrees with $h(1\times f)$ gep near $\eta(I\times B^k\times rB^n)$, so it is PL there.

(62)

$$\begin{tikzcd}
I\times B^k\times B^n \arrow[r, "\eta", "1 \text{ near } J"'] \arrow[d, "\sigma", "\text{PL embedding}"'] \arrow[dr, phantom, "\text{commutes near } I\times B^k\times B^n"] & I\times B^k\times B^n \arrow[d, "\sigma''"] \\
I\times B^k\times\mathbb{R}^n \arrow[r, "h", "\text{PL near } J"'] & M
\end{tikzcd}$$

$h\sigma\eta^{-1}$ is PL near $\eta(I\times B^k\times rB$



$W=I\times B^k\times B^n-\eta(I\times B^k\times rB^n)$ is a PL manifold (since an open subset of PL manifold)

If $n\geq 3$ W is simply-connected at infinity, so if $n\geq 3$ Browder-Levine-Livesay thm (5.2(B)) implies that

$$W\cong \text{open subset of compact manifold}$$

If $n\leq 2$, same result, using instead Siebenmann's thesis. Follows that $\eta(I\times B^k\times rB^n)$ has a neighbourhood which is a compact PL manifold such that

$$\partial N\subset \overline{I\times B^k\times B^n-N} \quad \text{is a hty. equivalence}$$

Now s-cobordism theorem (Prop 5.5) $\Rightarrow \overline{I\times B^k\times B^n-N}\underset{PL}{\cong}\overline{I\times B^k\times B^n-I\times B^k\times rB^n}$

Now follows that $\exists\,\sigma'': I\times B^k\times B^n\longrightarrow M$, a PL embedding such that $\sigma''\eta=h\sigma$ near $I\times B^k\times rB^n$ (regard $\overline{I\times B^k\times B^n-N}$ as a collar of $\mathrm{Fr}\,N$).

(63) Let R_t be isotopy from η to 1 rel $]$.

Define $S_t : I \times B^k \times \mathbb{R}^n \longrightarrow M$ by

$$S_t(x) = \begin{cases} \sigma'' R_t \eta^{-1} (\sigma'')^{-1} h(x) & h(x) \in \operatorname{im} \sigma'' \\ h(x) & h(x) \notin \operatorname{im} \sigma'' \end{cases}$$

Then $S_0 = h$, $S_1|_{I \times B^k \times sB^n} = \sigma'' R_1 \eta^{-1} \eta \sigma^{-1} |$
$(\not\subseteq \sigma(I \times B^k \times rB^n))$

$$= \sigma'' \sigma^{-1} |_{I \times B^k \times sB^n} \longrightarrow M$$

which is PL.

$S_t = h$ on $]$ and also outside $h^{-1}(\text{image } \sigma'')$ which is compact.

$\therefore S_t = h$ on $]$ and outside $I \times B^k \times RB^n$ for some $R \gg 0$.

Trivial to replace S_t by isotopy H_t satisfying 1) – 3).

$\square$

Theorem 5.6 Let C, D be closed subsets of $\mathbb{R}^n$ and let U be an open neighbourhood of C. Let $\mathcal{F}$ be a PL structure on $U \times I \subset \mathbb{R}^n \times I$ which agrees with the standard PL structure near $(U \cap D) \times I$, and near $U \times 0$. If $n \geq 6$, then there is an isotopy H_t of $\mathbb{R}^n \times I$ such that:

1) $H_0 = 1_{\mathbb{R}^n \times I}$

2) $H_1 : (U \times I, \text{standard}) \longrightarrow (U \times I, \mathcal{F})$ is PL near $C \times I$

3) $H_t = 1$ near $(D \cup (\mathbb{R}^n - U)) \times I$ and near $\mathbb{R}^n \times 0$.

(64) Proof: If C, D are compact, this is deduced from 5.4 exactly as 4.7 was deduced from 4.4. For the general case, let $C_i = C \cap iB^n$, $U_i = U \cap (i+1) \operatorname{int} B^n$, $D_i = D \cap (i+1) B^n$. Suppose inductively $H^{(i)}$ satisfies 1) – 3) w.r.t. C_i, D_i, U_i.

Let $\mathcal{F}_i = (H_1^{(i)})^{-1}(\mathcal{F})$: this is a PL structure on $U \times I$ which agrees with the standard PL structure ~~as~~ near $(C_i \times I) \cup (D_i \cup (\mathbb{R}^n - U_i)) \times I$ and near $U \times 0$. Now apply compact case to get isotopy H'_t satisfying 1) – 3) w.r.t. $\overline{C_{i+1} - C_i}$, $U_{i+1} - \overline{U_{i-2}}$, $C_i \cup D_{i+1}$, $\mathcal{F}_i$. Then

$H_t^{(i+1)} = H_t^{(i)} H'_t$ satisfies (1) – (3) w.r.t. $C_{i+1}, U_{i+1}, D_{i+1}, \mathcal{F}$.

Since $H'_t = 1$ on ~~U_{i-2}~~ $(i-1)B^n$, $H_t^{(i+1)} = H^i_{(t)}$ on $(i-1)B^n$.

Now take

$$H_t = \lim_{i \to \infty} H_t^{(i)}$$: this satisfies 1) – 3) w.r.t. $C, D, U, \mathcal{F}$.

$\square$

Theorem 5.7 (Product structure theorem). Let M^n be a topological manifold; let $C \subseteq M$ be a closed subset, and let U be an open neighbourhood of C. Let $\mathcal{F}_0$ be a PL structure on U, and let $\mathcal{G}$ be a PL structure on $M\times\mathbb{R}^k$, such that $\mathcal{G}$ agrees with $\mathcal{F}_0\times\mathbb{R}^k$ on $U\times\mathbb{R}^k$. If $n\geqslant 6$, then $\exists$ PL structure $\mathcal{F}$ on M, agreeing with $\mathcal{F}_0$ on C, and a PL homeo $(M\times\mathbb{R}^k, \mathcal{F}\times\mathbb{R}^k)\longrightarrow(M\times\mathbb{R}^k, \mathcal{G})$ which is isotopic to 1 by an isotopy fixing a neighbourhood of $C\times\mathbb{R}^k$.

□ proof below

Definition 5.8 PL structures $\mathcal{F}_1, \mathcal{F}_2$ on M are isotopic if there is a PL homeo $h:(M,\mathcal{F}_1)\to(M,\mathcal{F}_2)$ which is isotopic to 1.

Let $PL(M)$ be the set of isotopy classes of PL structures on M.

Corollary 5.9 If $\dim M\geqslant 6$, the natural map

$$PL(M)\longrightarrow PL(M\times\mathbb{R}^k)$$

is a bijection. In particular, if $M\times\mathbb{R}^k$ has a PL structure, and $\dim M\geqslant 6$, then M has a PL structure.

□

Lemma 5.10 Any two PL structures on $\mathbb{R}^n$ ($n\geqslant$ are isotopic.

Proof: Let $\mathcal{F}$ be a PL structure on $\mathbb{R}^n$. By Prop 5.2 (B) $(\mathbb{R}^n,\mathcal{F})$ is PL homeo to $\operatorname{Int} W$, where W is compact PL manifold $\partial W \simeq S^{n-1}$, so by 5.2 (A) $\partial W \underset{PL}{\cong} S^{n-1}$

W is contractible, so by 5.2 (A), $W \underset{PL}{\cong} B^n$.

$W\cup_\partial B^n \cong S^n$. So $\exists$ PL homeo

$$h:\mathbb{R}^n\longrightarrow \operatorname{Int} B^n \longrightarrow \operatorname{Int} W\longrightarrow(\mathbb{R}^n,\mathcal{F})$$

Can assume h is orientation-preserving.

Must prove h isotopic to 1.

Let $R>r>0$ be chosen so that

$$h(rB^n)\subset \operatorname{Int} h(RB^n)$$

By annulus theorem 5.3, $\exists$ homeomorphism

$$f: RB^n-\operatorname{Int} rB^n\longrightarrow \cancel{RB^n} \; RB^n - h(\operatorname{Int} rB^n)$$

with $f|_{\partial(RB^n)}=1$. Since h is orientation-preserving, and using the proof of 5.3 we can choose f so that $f=h$ on $\partial(rB^n)$

Extend f over $\mathbb{R}^n$ by $f(x)=\begin{cases} x & \|x\|\geqslant R\\ hx & \|x\|\leqslant r\end{cases}$

(67) Since $f = 1$ outside RB^n, f isotopic to 1, so h isotopic to $f^{-1}h$.

Since $f^{-1}h = 1$ in rB^n, $f^{-1}h$ is isotopic to 1.

$\therefore$ h isotopic to 1 as required. $\square$

Proof of Thm 5.7 Clearly sufficient to prove for case $k=1$. Assume first $M = \mathbb{R}^n$.

$\mathcal{G}$ = PL structure on $\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$. By 5.10, $\exists$ isotopy H_t s.t. $H_1 : \mathbb{R}^{n+1} \longrightarrow \underbrace{\mathbb{R}^{n+1}, \mathcal{G}}_{\text{(together)}}$ is PL and $H_t = 1$ for $t \leq \frac{1}{4}$. H defines homeomorphism

$H : \mathbb{R}^n \times \mathbb{R} \times I \longrightarrow \mathbb{R}^n \times \mathbb{R} \times I$ $\quad (x,t) \longmapsto (H_t(x), t)$

Let $\mathcal{H} = H(\text{std. PL structure})$. Then $\mathcal{H}$ agrees with standard structure near $\mathbb{R}^n \times \mathbb{R} \times 0$, and with $\mathcal{G}$ on $\mathbb{R}^n \times \mathbb{R} \times 1$

Apply Theorem 5.6 to $\mathbb{R}^n \times \mathbb{R} \times I$ with $C, U, D, \mathcal{F}$ replaced by $\mathbb{R}^n \times (-\infty, 0]$, $\mathbb{R}^n \times (-\infty, \frac{1}{2})$, $\emptyset$, $\mathcal{H}|_{U \times I}$.

(68) We obtain isotopy $F_t : \mathbb{R}^n \times \mathbb{R} \times 1 \circlearrowleft$ s.t. $F_0 = 1$, $F_t = 1$ outside $\mathbb{R}^n \times (-\infty, \frac{1}{2}) \times 1$ and $F_1 : \mathbb{R}^n \times (-\infty, \frac{1}{2}) \times 1, \text{std} \longrightarrow (\mathbb{R}^n \times (-\infty, \frac{1}{2}) \times 1, \mathcal{H})$ is PL near $\mathbb{R}^n \times (-\infty, 0] \times 1$.

Let $\mathcal{G}' = F_1^{-1}(\mathcal{G})$ - PL structure on $\mathbb{R}^n \times \mathbb{R}$

Then $\mathcal{G}'$ agrees with $\mathcal{G}$ near $\mathbb{R}^n \times [1, \infty)$ and $\mathcal{G}'$ agrees with standard structure near $\mathbb{R}^n \times (-\infty, 0]$.

$\mathbb{R}^n \times 0$ is a PL submanifold of $\mathcal{G}'$

$U \times 1$ is a PL submanifold of $\mathcal{G}'$

$\therefore$ $\mathcal{G}'$ induces a PL structure on $U \times I$

$\mathcal{G}'$ = standard structure near $U \times 0$

Apply Theorem 5.6 to $C, U, \phi, \mathcal{G}'|_{U \times I}$ to obtain isotopy G_t of $\mathbb{R}^n \times \mathbb{R}$ such that

$G_1 : (U \times I, \text{std.}) \longrightarrow (U \times I, \mathcal{G}')$ is PL near $C \times I$

G_t is 1 near $\mathbb{R}^n \times 0$.

Let $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be defined by $(g(x), 1) = G_1(x, 1)$

Let $\mathcal{G}'' = (g \times 1) G_1^{-1}(\mathcal{G}')$

(69)

Near $C\times I$, $\mathcal{G}''$ agrees with $(g\times 1)$(standard structure) $= \mathcal{F}_g\times I$ (in nbhd of $C\times I$)

Define $\mathcal{F} = \mathcal{G}''|_{\mathbb{R}^n\times 0}$.

$\mathcal{F}$ agrees with $\mathcal{F}_0$ near C

Remains constant isotopy (rel $C\times\mathbb{R}$) from $\mathcal{F}\times\mathbb{R}$ to $\mathcal{G}$.

Choose a PL isotopy (of embeddings) $j_t:\mathbb{R}\longrightarrow\mathbb{R}$ s.t. $j_t = 1$ $(t\leq\frac{1}{4})$ and $j_1(\mathbb{R})\subset(1,\infty)$

$J: \mathbb{R}^n\times\mathbb{R}\times I\longrightarrow\mathbb{R}^n\times\mathbb{R}\times I$ defined by $J(x,y,t) = (x, j_t(y), t)$.

Then PL structure $J^{-1}(\mathcal{G}''\times I)$ agrees with $\mathcal{G}''\times 0$ on $\mathbb{R}^n\times\mathbb{R}\times 0$ and agrees with $\mathcal{F}_0\times\mathbb{R}\times I$ near $C\times\mathbb{R}\times I$.

Apply theorem 5.6 (using fact that $\mathcal{G}''$ isotopic to standard structure by Lemma 5.10) to obtain isotopy from $\mathcal{G}''$ to $J^{-1}(\mathcal{G}''\times 1)$, fixed near $C\times\mathbb{R}$. We have $J^{-1}(\mathcal{G}''\times 1) = J^{-1}(\mathcal{G}\times 1)$

(70)

(since $\mathcal{G}''=\mathcal{G}$ on $\mathbb{R}^n\times(1,\infty)$) and similarly $\mathcal{G}$ is isotopic to $J^{-1}(\mathcal{G}\times 1)$ (fixed near $C\times\mathbb{R}$)

$\therefore$ $\mathcal{G}, \mathcal{G}''$ isotopic (relative to neighbourhood of $C\times\mathbb{R}$)

Similarly, $\mathcal{G}'', \mathcal{F}\times\mathbb{R}^n$ isotopic fixing nbhd of $C\times\mathbb{R}$

$\therefore$ $\mathcal{G}, \mathcal{F}\times\mathbb{R}$ ——— "

For general M: with $\partial M=\emptyset$.

(W.l.o.g. connected)

We know that M is metrizable $\Rightarrow$ M is 2nd countable

So $M = \bigcup_{i=1}^{\infty} f_i(B^n)$ where $f_i:\mathbb{R}^n\longrightarrow M$ is an embedding

Let $C_i = C\cup f_1(B^n)\cup\ldots\cup f_i(B^n)$.

Suppose inductively we have PL structure $\mathcal{F}_{i-1}$ on nbhd of C_{i-1} in M, extending $\mathcal{F}_0$ and PL structure $\mathcal{G}_{i-1}$ on $M\times\mathbb{R}$ extending $\mathcal{F}_{i-1}\times\mathbb{R}$ and isotopic to $\mathcal{G}$ by isotopy fixed near $C\times\mathbb{R}$.

Apply the result for $M=\mathbb{R}^n$ to $\mathcal{F}' = f_i^{-1}(\mathcal{F}_{i-1})$ (near $C' = f_i^{-1}(C_{i-1})$) and $(f_i\times 1)^{-1}(\mathcal{G}_{i-1}) = \mathcal{G}'$.

We obtain a PL structure $\mathcal{F}''$ on $\mathbb{R}^n$ ($=\mathcal{F}'$ near C')

(71) and isotopy H_t of $\mathbb{R}^n \times \mathbb{R}$ with $H_t = 1$ $(t \leq \frac{1}{4})$ and $H_1^{-1}(\mathcal{G}') = \mathcal{F}'' \times \mathbb{R}$, and H_t fixes a nbhd of C'.

H defines homeo $H : \mathbb{R}^n \times \mathbb{R} \times I \circlearrowleft$:
let $\mathcal{H} = H^{-1}(\mathcal{G}' \times I)$. $\mathcal{H}$ agrees with $\mathcal{G}'$ near $\mathbb{R}^n \times \mathbb{R} \times 0$, with $\mathcal{F}'' \times \mathbb{R}$ on $\mathbb{R}^n \times \mathbb{R} \times 1$, and near $C' \times \mathbb{R} \times I$. Apply Theorem 5.6 to this: replace $C, U, D, \mathcal{F}$ by $B^n \times \mathbb{R}$, $(\mathrm{Int}\, 2B^n \times \mathbb{R})$, $C' \times \mathbb{R}$, $\mathcal{H}$ to obtain PL structure $\mathcal{G}''$ on $\mathbb{R}^n \times \mathbb{R}$ which agrees with $\mathcal{F}'' \times \mathbb{R}$ near $(C' \cup B^n) \times \mathbb{R}$ and which is isotopic to $\mathcal{G}'$ rel $(C' \cup (\mathbb{R}^n - \mathrm{Int}\, 2B^n)) \times \mathbb{R}$.

Define $\mathcal{F}_i = \mathcal{F}_{i-1} \cup f_i(\mathcal{F}'')$ and extend $(f_i \times 1)(\mathcal{G}'')$ to structure $\mathcal{G}_i$ on $M \times \mathbb{R}$ agreeing with $\mathcal{G}_{i-1}$ off $f_i(\mathbb{R}^n) \times \mathbb{R}$.

Then $\mathcal{G}_i$ agrees with $\mathcal{F}_i \times \mathbb{R}$ near $C_i \times \mathbb{R}$ and $\mathcal{G}_i$ isotopic to $\mathcal{G}_{i-1}$ fixing nbhd of $C_{i-1} \times \mathbb{R}$.

(72) So $\mathcal{F}_i = \mathcal{F}_{i-1}$ near C_{i-1}.

Since $\mathcal{F}_i = \mathcal{F}_{i-1}$ near C_{i-1}, $\exists$ PL structure $\mathcal{F}$ on M agreeing with $\mathcal{F}_i$ near C_i. $\mathcal{G}$ agrees with $\mathcal{F} \times \mathbb{R}$ near $C \times \mathbb{R}$, $\mathcal{F}$ agrees with $\mathcal{F}_0$ near C.
Since $\mathcal{G}_i$ isotopic to $\mathcal{G}_{i-1}$ (fixing nbhd of $C_{i-1} \times \mathbb{R}$). Hence all isotopies can be pieced together to obtain isotopy of $\mathcal{F} \times \mathbb{R}$ to $\mathcal{G}$, fixing nbhd of $C \times \mathbb{R}$. This proves product theorem when M has no boundary.

If M has non-empty boundary ∂M, then apply theorem for M unbounded to ∂M, and then to $\mathrm{Int}\, M$, using a collaring argument. Seem to need $\dim M \geq 7$ (to ensure $\dim \partial M \geq 6$) $\square$

(In fact theorem can be proved for all unbounded 5-manifolds and all 6-manifolds).

(73). Applications: If M is a topological manifold, we can embed M in $\mathbb{R}^N$ with a nbhd E which fibers over M, i.e. $\exists$ map $p: E \longrightarrow M$ which is locally the projection of product, with fibre $\mathbb{R}^n$ (structural group $\mathcal{H}(\mathbb{R}^n) = Top_n$)
Let $\nu = (p: E \to M)$

A necessary condition for M to have a PL structure is that ν come from a PL bundle over M.

Also sufficient (if $\dim M \geqslant 6$)

$E(\nu)$ is open subset of $\mathbb{R}^N$ so that it inherits a PL structure. Suppose $\exists$ PL bundle ξ over $E(\nu)$ which is equivalent ~~to~~ as topological bundle to ν. $\exists$ PL bundle η over $E(\nu)$ s.t. $\xi \oplus \eta$ is trivial. Then total space $E(\eta)$ is homeomorphic to $M \times \mathbb{R}^k$ and has a PL structure.

By product structure theorem, M has a PL structure.

(74) $\exists$ classifying space $BTop_n$ classifying such topological ~~bdl~~ bundles by $[M, BTop_n]$: n immaterial so take $BTop = \bigcup_{n=1}^{\infty} BTop_n$.

Similarly for BPL_n, BPL. $\exists$ natural map $BPL_n \to BTop_n$ (forgetful)

$\therefore$ $M^{n \geq 6}$ has a PL structure if the map $\nu: M \longrightarrow BTop$ factors (up to homotopy) as

$$\begin{tikzcd} & M \arrow[dl, dotted] \arrow[d, "\nu"] \\ BPL \arrow[r] & BTop \end{tikzcd}$$

$\therefore$ M has a PL structure iff the classifying map of the stable normal bundle ν of M lies in the image of $[M; BPL] \longrightarrow [M, BTop]$.

To show that $PL \not\equiv Top$:

Let k be an integer, and $p_k: T^n \longrightarrow T^n$ be induced by $\mathbb{R}^n \longrightarrow \mathbb{R}^n$; $x \longmapsto kx$

Then p_k is a k^n-fold covering (fibre bundle with discrete fibre of k^n pts)

$\exists$ homeomorphism $h_k : T^n \to T^n$ s.t.

$$\begin{tikzcd} T^n \arrow[r, dashed, "h_k"] \arrow[d, "p_k"'] & T^n \arrow[d, "p_k"] \\ T^n \arrow[r, "h"] & T^n \end{tikzcd}$$

for any given homeomorphism $h : T^n \to T^n$.

There are k^n such homeomorphisms. Since all covering translations of $T^n \xrightarrow{p_k} T^n$ are isotopic to 1, any 2 choices for h_k are isotopic.

__Theorem 5.11__ If $h : T^n \to T^n$ is a homeomorphism homotopic to 1, then h_k is topologically isotopic to 1 for sufficiently large k.

__Proof__: First isotop h until $h(0) = 0$ (where $0 = e(0) \in T^n$)

Choose h_k so that $h_k(0) = 0$. Let $\tilde{h}_k : \mathbb{R}^n \to \mathbb{R}^n$ be homeo such that $e\tilde{h}_k = h_k e$ and $\tilde{h}_k(0) = 0$.

Since $h \simeq 1$, ~~$h_k \simeq 1$~~ so $\tilde{h}_1 = \tilde{h}$ is bounded.

$\tilde{h}_k(x) = \frac{1}{k}\tilde{h}(x)$ because $p_k e \tilde{h}_k = p_k h_k e = h p_k e$, $\tilde{h}_k(0) = 0$ & these characterise $\tilde{h}_k$.

$$\sup_{x \in \mathbb{R}^n} d(x, \tilde{h}_k(x)) = \frac{1}{k}\left(\sup_{x \in \mathbb{R}^n} d(x, \tilde{h}(x))\right) \to 0 \text{ as } k \to \infty$$

So $\sup_{y \in T^n} d(y, h_k(y)) \to 0$ as $k \to \infty$

But $\mathcal{H}(T^n)$ is locally contractible (by Thm 4.8).

$\therefore$ If k large enough, h_k isotopic to 1.

But behaviour is different in PL case:

~~Theorem~~ __Proposition 5.12__ (Wall) Let $n \geq 5$ $\exists$ PL homeomorphism $h : T^n \to T^n$ s.t. $h \simeq 1$ and h_k is not PL isotopic to 1 for any odd k.

__Proof__ (sketch)

$$\begin{tikzcd} \left\{ \begin{matrix} \text{PL isotopy classes} \\ \text{of PL homeos of } T^n \end{matrix} \right\} \arrow[r, phantom, "\cong"] & H^2(T^n; \mathbb{Z}_2) \arrow[d, "p_k^!"', "k \text{ odd}"] \\ & H^2(T^n; \mathbb{Z}_2) \end{tikzcd}$$

~~Exercise Show that, if h is as in 5.12~~, then

(7.7).

Exercise Show that, if $h: T^n \longrightarrow T^n$ is PL and (topologically) isotopic to 1, but not PL isotopic to 1, then $T^n \times I / (x,0) \sim (h(x),1)$ is topologically homeomorphic to T^{n+1}, but not PL homeomorphic to T^{n+1}.

$\square$ $\square$ ...