Contribution to the Encyclopædia of Mathematics, Kluwer (2000)

The mapping torus of an automorphism of a manifold

The mapping torus of a self map $h : F \to F$ is the identification space

$$T(h) \;=\; F \times [0,1]/\{(x,0) \sim (h(x),1) \,|\, x \in F\}$$

which is equipped with a canonical map

$$p \;:\; T(h) \;\to\; S^1 \;=\; [0,1]/\{0 \sim 1\}\;;\; (x,t) \;\to\; t\;.$$

If F is a closed n-dimensional manifold and $h : F \to F$ is an automorphism then $T(h)$ is a closed $(n+1)$-dimensional manifold such that p is the projection of a fibre bundle with fibre F and monodromy h. If F is an n-dimensional manifold with boundary and $h : F \to F$ is an automorphism such that $h| = 1 : \partial F \to \partial F$ then $T(h)$ is an $(n+1)$-dimensional manifold with boundary $\partial T(h) = \partial F \times S^1$, and the union

$$t(h) \;=\; T(h) \cup_{\partial T(h)} \partial F \times D^2$$

is a closed $(n+1)$-dimensional manifold, called an open book. It is important to know when manifolds are fibre bundles over S^1 and open books, for in those cases the classification of $(n+1)$-dimensional manifolds is reduced to the classification of automorphisms of n-dimensional manifolds.

A codimension 2 submanifold $K^n \subset M^{n+2}$ is fibred if it has a neighbourhood $K \times D^2 \subset M$ such that the exterior $X = \mathrm{cl.}(M \backslash K \times D^2)$ is a mapping torus, i.e. if $X = t(h)$ is an open book for some automorphism $h : F \to F$ of a codimension 1 submanifold $F^{n+1} \subset M$ with $\partial F = K$ (a Seifert surface). Fibred knots $S^n \subset S^{n+2}$ and fibred links $\cup S^n \subset S^{n+2}$ have particularly strong geometric and algebraic properties.

In 1923 J.W. Alexander used geometry to prove that every closed 3-dimensional manifold M^3 is an open book, that is there exists a fibred link $\cup S^1 \subset M$, generalizing the Heegaard splitting.

Fibred knots $S^n \subset S^{n+2}$ came to prominence in the 1960's with the influential work of J. Milnor on singular points of complex hypersurfaces, and with the examples of E. Brieskorn realizing the exotic spheres as links of singular points.

Connected infinite cyclic covers $\overline{M}$ of a connected space M are in one-one correspondence with expressions of the fundamental group $\pi_1(M)$ as a group extension

$$\mathbb{E} \;:\; 1 \to \pi_1(\overline{M}) \to \pi_1(M) \to \mathbb{Z} \to \{1\}\;,$$

and also with the homotopy classes of maps $p : M \to S^1$ inducing surjections $p_* : \pi_1(M) \to \pi_1(S^1) = \mathbb{Z}$. If $p : M \to S^1$ is the projection of a fibre bundle with M compact the noncompact space $\overline{M}$ is homotopy equivalent to the fibre F, which is compact, so that the fundamental group $\pi_1(\overline{M}) = \pi_1(F)$ and the homology groups $H_*(\overline{M}) = H_*(F)$ are finitely generated.

In 1962 J. Stallings used group theory to prove that if M is an irreducible closed 3-dimensional manifold with $\pi_1(M) \neq \mathbb{Z}_2$ and with an extension $\mathbb{E}$ such that $\pi_1(\overline{M})$ is finitely generated then M is a fibre bundle over S^1, with $M = T(h)$ for some automorphism $h : F \to F$ of a surface $F^2 \subset M$. In 1964 W. Browder and J. Levine used simply-connected surgery to prove that for $n \geq 6$ every closed n-dimensional manifold M with $\pi_1(M) = \mathbb{Z}$ and $H_*(\overline{M})$ finitely generated is a fibre bundle over S^1. In 1984 M. Kreck used this type of surgery to compute the bordism groups Δ_* of automorphisms of high-dimensional manifolds and to evaluate the mapping torus map

$$T \ : \ \Delta_n \to \Omega_{n+1}(S^1) \ ; \ (F^n, h : F \to F) \to T(h)$$

to the ordinary bordism over S^1.

A band is a compact manifold M with a connected infinite cyclic cover $\overline{M}$ which is finitely dominated, i.e. such that there exists a finite CW complex K with maps $f : \overline{M} \to K$, $g : K \to \overline{M}$ and a homotopy $gf \simeq 1 : \overline{M} \to \overline{M}$. In 1968 F.T. Farrell used non-simply-connected surgery theory to prove that for $n \geq 6$ a PL (or differentiable) n-dimensional manifold band M is a fibre bundle over S^1 if and only if a Whitehead torsion obstruction $\Phi(M) \in Wh(\pi_1(M))$ is 0. The theorem was important in the structure theory of high-dimensional topological manifolds, and in 1970 was extended to topological manifolds by L. Siebenmann. There is also a version for Hilbert cube manifolds, obtained in 1974 by T.A. Chapman and Siebenmann. The fibering obstruction $\Phi(M)$ for finite-dimensional M measures the difference between the intrinsic simple homotopy type of M given by a handlebody decomposition and the extrinsic simple homotopy type given by $M \simeq T(\zeta)$ with $\zeta : \overline{M} \to \overline{M}$ a generating covering translation.

In 1972 H.E. Winkelnkemper used surgery to prove that for $n \geq 7$ a simply-connected n-dimensional manifold M is an open book if and only if the signature of M is 0. In 1977 T. Lawson used non-simply-connected surgery to prove that for odd $n \geq 7$ every n-dimensional manifold M is an open book. In 1979 F. Quinn used non-simply-connected surgery to prove that for even $n \geq 6$ an n-dimensional manifold M is an open book if and only if an obstruction in the asymmetric Witt group of $\mathbb{Z}[\pi_1(M)]$ vanishes, generalizing the Wall surgery obstruction.

For a recent account of fibre bundles over S^1 and open books see the book "High-dimensional knot theory" (Springer, 1998) by the reviewer.

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