Differential Topology

C. T. C. Wall

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Contents

Introduction

These notes are based on a seminar held in Cambridge 1960-61. In writing up, it has seemed desirable to elaborate the foundations considerably beyond the point form which the lectures started, and the notes have expanded accordingly; this is only the first set. It is divided three parts:

- Part 0 Analytical Foundations
- Part I Geometrical Foundations
- Part II Theorems of Transversality and General Position

(No index is included since numeration and pagination are by chapters.)

We hope to have given a thorough treatment of the basic theorems of use in investigating smooth manifolds; the only others to my knowledge are a paper by J. Cerf (Bulletin de Société Mathématique de France 89, pp227-380 (1961)) and a forthcoming book by S. Lang (on unbounded manifolds only). It is intended that subsequent parts of these notes shall be as follows: imbeddings and immersions, cobordism theory, the h-cobordism theorem, and surgery; however, this is somewhat optimistic.

It is perhaps appropriate to comment here on a few points which were only noticed when notes were typed out. Part 0, 4.3 (the Implicit Function Theorem) is not needed; a proof can be given as in I.2.5. Proofs of 0, 4.1 and 0, 4.5 can be found in any good book on analysis. The proof of I, 4.8 is cooked: I should have extended the method of proof of I, 4.4. The proofs in I, 5 of uniqueness of tubular neighbourhoods can be used to give a local piercing together, and hence prove existence also: this avoids the difficulties in I, 6.2, and is the method adopted by Cerf and Lang. I have used a more direct geometrical construction by preference; the other method is, however, stronger, and removes the restriction to compact submanifolds, thus answering, for example, the problem of I, 7.2. By an oversight, the existence part of the proof of I, 6.7 was omitted - it is very simple, the reader will easily supply it for himself.

I am indebted to all the Cambridge topology research students of last year for participating in the seminar; in particular to P. Baxandall for taking notes on the first 6 seminars, and to Steve Gersten for doing the rest, and for considerable assistance in writing up.

Notations, etc.

We assume known a certain amount of analysis, and a few terms and results from analytical topology - for example,

- "φ" denotes a metric, and a paracompact space is defined by the property that any open covering admits a locally finite refinement.
- The word "smooth" is always used to mean "infinitely differentiable", i.e., C^∞.
- We use $\mathbb{R}$ to denote the real numbers, $\mathbb{R}^n$ for the vector space of n-tuples, with is usual metric and topological structure, $\mathbb{R}^n_+$, $\mathbb{R}^n_{++}$ for the subsets with the first, or first two terms non-negative.
- For $x \in \mathbb{R}^n$, $|x|$ is the root square sum of the terms, and $U(x, r) = \{y \colon |y - x| < r\}$.
- $\mathbf{GL}_n(\mathbb{R})$ is the group of non-singular linear transformation of $\mathbb{R}^n$, with subgroups $\mathbf{GL}^n_+(\mathbb{R})$, (with positive determinant), the orthogonal group $\mathbf{O}_n$ (preserving the metric $|x|$), and $\mathbf{SO}_n$, their intersection.
- The interval I is the subset $0 \leqq x \leqq 1$ of $\mathbb{R}$, and D^n, S^{n-1} are the subsets $|x| \leqq 1$, $|x| = 1$ of $\mathbb{R}^n$.
- We denote set membership by $\in$, and set inclusion by $\subset$.
- The restriction of a map f to a subset X of the domain is $f|X$.
- Composition of maps is (usually) denoted by a circle, as $f \circ g$, and is written in the illogical order.
- The image of a map of f is $\mathrm{Im} f$.
- If X, Y are sets, $X \times Y$ is the set of pairs $\{(x, y) \colon x \in X, y \in Y\}$, and $\Delta(X)$ is the diagonal subset of $X \times X$, with pairs $\{(x, x) \colon x \in X\}$.
- Finally, the conclusion of a proof is signaled by $\square$.

Part 0

Analytical Foundations

Chapter 1

Definitions

Definition 1.1. A *smooth n-manifold* M^n is a paracompact Hausdorff space with a family $\mathcal{F} = \mathcal{F}_M$ of continuous real-valued functions defined on M and satisfying the following conditions:

i) $\mathcal{F}$ is local. If $f\colon M \to \mathbb{R}$ is such that each point of M has a neighbourhood in which f agrees with a function of $\mathcal{F}$, then $f \in \mathcal{F}$.

ii) $\mathcal{F}$ is differentiably closed. If $f_1, \ldots, f_k \in \mathcal{F}$, and F is a smooth function on $\mathbb{R}^n$, then $F(f_1, \ldots, f_k) \in \mathcal{F}$.

iii) $(M, \mathcal{F})$ is locally Euclidean. For each point $P \in M$, there are m functions $f_1, \ldots, f_m \in \mathcal{F}$ such that $Q \mapsto (f_1(Q), \cdots, f_m(Q))$ gives a homeomorphism of a neighbourhood U of P onto an open subset V of $\mathbb{R}^m$. Every function $f \in \mathcal{F}$ coincides on U with $F(f_1, \ldots, f_m)$, where F is a smooth function on V.

We call functions $f \in \mathcal{F}$ *smooth functions* of M, and the mapping defined in iii) (or, by abuse of language, the set U) a *coordinate neighbourhood*, or *C.N.* of P. It follows from ii) that sums, products, and constant multiples of smooth functions are also smooth.

The first tool we need, to work with the above definition, is a bump function. Define first a function ψ on $\mathbb{R}^1$ by:

$$\psi(x) = \begin{cases} \exp(1/(x(x-1))) & \text{if} \quad 0 \leqq x \leqq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then ψ is smooth, non-negative, and differs from zero when $0 < x < 1$.

Definition 1.2. The *bump function* $B_P(x)$ is now given by

$$B_P(x) = \int_0^y \psi(t)dt \Big/ \int_0^1 \psi(t)dt.$$

Since $\psi(x)$ is smooth, so is $B_P(x)$. Also

$$\begin{aligned} B_P(x) &= 0 \quad \text{if} \quad x \leqq 0, \\ 0 < B_P(x) &< 1 \quad \text{if} \quad 0 < x < 1, \quad \text{and} \\ B_P(x) &= 1 \quad \text{if} \quad x \geqq 1 \end{aligned}$$

These are the essential properties of the bump function; any other function with them would serve the same purpose. We now have

Proposition 1.3. *Let $\varphi\colon U \to V$ be a C.N. of $P \in M$, and let F be a smooth function on V. Then there is a function $f \in \mathcal{F}$, agreeing with $F \circ \varphi$ in a neighbourhood of P, and zero outside U.*

Proof. Without loss of generality, let $\varphi(P) = 0$. Since V is a neighbourhood of 0, we can find $r > 0$ with $U(0, 3r) \subset V$. Define $\Phi(x) = B_P(2 - |x|/r)$ then $\Phi(x) = 1$ for $|x| \leqq r$, $\Phi(x) = 0$ for $|x| \geqq 2r$, and Φ is a smooth function on $\mathbb{R}^m$, hence also on V, since B_P is smooth, and $|x|$ is smooth except at 0. Then $f\Phi$ is also smooth on V, and $F(x)\Phi(x) = 0$ if $|x| \geqq 2r$. We define a function f on M by:

$$f(P) = \begin{cases} F\Phi \circ \varphi(P) & \text{if} \quad P \in M \\ 0 & \text{otherwise.} \end{cases}$$

Then, by condition ii), $f \in \mathcal{F}$, and f agrees with $f \circ \varphi$ in $\varphi^{-1}\{x\colon |x| \leqq r\}$. □

This proposition enables us to observe that the above definition of smooth manifold coincides with the definition in terms of an open covering $\{U_\alpha\}$ of M, each U_α provided with a homeomorphism φ_α onto an open subset of $\mathbb{R}^m$, such that in the intersection $U_\alpha \cap U_\beta$ we have a smooth change of coordinates. Indeed, the only real difference between this definition and Definition 1.1 is that Definition 1.1 requires φ_α to be defined by functions which extend to smooth functions on the whole M. But since the proof of Proposition 1.3 is equally valid for the other definition, we see that any locally smooth functions, provided we allow their range to be slightly restricted, extend smoothly to all M.

We now give some simple examples of smooth manifolds.

1. The empty set is a smooth m-manifold (the definition is vacuously satisfied).

2. $\mathbb{R}^m$, with smooth functions taken in the ordinary sense, is a smooth m-manifold. Condition i) is trivial, ii) follows from the rule for differentiating a function of a function, and for iii), since the coordinate functions are smooth, we take the identity map.

3. The discrete union of an arbitrary set of smooth m-manifolds is another. Define a function to be smooth if the induced function on each summand is so; the conditions are then all trivial.

4. Let M be a smooth m-manifold, O an open subset of M. Write $\mathcal{G}_O$ for the restriction to O of functions of $\mathcal{F}_M$; $\mathcal{F}_O$ for the localisation of $\mathcal{G}_O$, i.e., the set of functions locally agreeing with a function of $\mathcal{G}_O$. Then it is clear, since O is open in M, that $(O, \mathcal{G}_O)$ satisfies conditions i), iii); $(O, \mathcal{F}_O)$ satisfies them and also condition ii). So in this way, the structure of smooth m-manifold on M naturally induces such a structure on O. We call O an *open submanifold* of M.

5. Let M^m, V^v be smooth manifolds. Then the topological product $N^{m+v} = M^m \times V^v$ has a natural structure of smooth manifold. For let π_1, π_2 denote projections on the factors. Then for $f \in \mathcal{F}_M$, $g \in \mathcal{F}_V$, we define $f \circ \pi_1$, $g \circ \pi_2$ to belong to $\mathcal{F}_N$; any smooth functions of a finite set of these; and any function locally agreeing with one of these functions. This definition ensures that conditions i) and ii) are satisfied. But so is iii), for it now follows that if $\varphi_1 \colon U_1 \to \mathbb{R}^m$, $\varphi_2 \colon U_2 \to \mathbb{R}^v$ are C.N.s in M and V, then $\varphi_1 \times \varphi_2 \colon U_1 \times U_2 \to \mathbb{R}^{m+v}$ can be taken as a C.N. in $M \times V$.

Definition 1.4. Let M^m, V^v be smooth manifolds. A mapping $\varphi \colon M \to V$ is called *smooth* if for each $f \in \mathcal{F}_V$, $f \circ \varphi \in \mathcal{F}_M$.

Note that in view of condition iii) this is equivalent to the requirement that each transformation of coordinates induced by φ between C.N.s in M and in V be smooth in the usual sense. However, the above definition is much more convenient.

Proposition 1.5. *If $\varphi_1 \colon M_1 \to M_2$ and $\varphi_2 \colon M_2 \to M_3$ are smooth, then so is $\varphi_2 \circ \varphi_1 \colon M_1 \to M_3$.*

Proof. If $f \in \mathcal{F}_3$, $f \circ \varphi_2 \in \mathcal{F}_2$, and so $f \circ \varphi_2 \circ \varphi_1 \in \mathcal{F}_1$ □

Proposition 1.6. *If O is an open submanifold of M, $i \colon O \subset M$ is smooth.*

Proof. If $f \in \mathcal{F}_M$, $f \circ i \in \mathcal{G}_O \subset \mathcal{F}_O$. □

These two propositions merely assert the consistency of our definitions. To conclude this chapter, we define the equivalence relation which classifies manifolds.

Definition 1.7. A 1 - 1 correspondence $\varphi \colon M^m \to V^m$ between two smooth manifolds is a *diffeomorphism* if both φ and φ^{-1} are smooth. M^m and V^m are called *diffeomorphic*.

Thus a diffeomorphism defines a 1 - 1 correspondence between the two manifolds, under which smooth functions correspond. Differential geometry and topology each consist of the study (from different points of view) of those properties of smooth manifolds which are invariant under diffeomorphisms.

Chapter 2

Analytic Topology

We collect in this chapter, for purpose of reference, most of the results from analytic topology of which we will later make use. The reader desiring continuity should read up to Proposition 2.11 and then go on to Chapter 3, referring back later when necessary. We first elucidate the conditions of paracompactness in Definition 1.1.

Theorem 2.1. *We can find a set of C.N.s* $\varphi_\alpha\colon U_\alpha \to U(0,3)$ *for* M^m *such that*

i) The sets $\varphi_\alpha^{-1}(U(0,1))$ *cover* M.

ii) Each $P \in M$ *has a neighbourhood in* M *which meets only a finite number of sets* U_α*, i.e., the* U_α *are locally finite. Moreover, the covering by the* U_α *may be chosen to refine any given covering of* M.

Proof. First take any set of C.N.s $\psi_\beta\colon O_\beta \to \mathbb{R}^m$ for M, such that the O_β cover M and refine the given covering. Since M is paracompact, there is a locally finite refinement $\{W_\beta\}$ of $\{O_\beta\}$, still covering M. If we now prove the result for W_β, the union of all such C.N.s for the various W_β satisfies the same conditions. But ψ_β defines a diffeomorphism of W_β on an open submanifold of $\mathbb{R}^m$. So we can suppose that M is an open submanifold of $\mathbb{R}^m$.

For each positive integer i, take all the open sets $U(x, 3\sqrt{m}/i)$ which are contained in M (actually, since we use Proposition 1.3 to say that a C.N. in W_β above is also one in M above, say: whose closures are contained in M), and such that ix has integral coordinates. Suppose $y \in M$; then some $U(y,\delta) \subset M$. Choose $i > 4\sqrt{m}/\delta$. Then some x with ix integral (i.e., $ix \in \mathbb{Z}$) is within a distance $\sqrt{m}/i$ of y, and $U(x, 3\sqrt{m}/i) \subset U(y, 4\sqrt{m}/i) \subset U(y,\delta) \subset M$. Thus the corresponding sets $U(x, \sqrt{m}/i)$ cover M. Delete any of these which is contained in another. Then the remaining ones still cover M. We say also, that the corresponding $U(x, 3\sqrt{m}/i)$ are locally finite. Now y has a neighbourhood of the form $U(x, \sqrt{m}/i)$; choose δ such that $U(y, 2\delta) \subset U(x, \sqrt{m}/i)$. Then if $j > 3\sqrt{m}/\delta$, and $U(z, 3\sqrt{m}/j)$ meets $U(y,\delta)$ it is contained in $U(y, 2\delta)$, and so in $U(x, \sqrt{m}/i)$, so it was one of the neighbourhoods which we discarded. Thus

$U(y, \delta)$ only meets sets $U(z, 3\sqrt{m}/j)$ with $j \leqq 3\sqrt{m}/\delta$, and hence only a finite number of such sets. □

Corollary 2.2. *Let f be a continuous positive function on M. Then we can find a smooth function g, with $0 < g(P) < f(P)$ for all $P \in M$.*

Proof. With the notation of the Theorem, choose $\delta_\alpha > 0$ less than the infimum of f on the compact set $\varphi_\alpha^{-1}(\bar{U}(0, 2)))$. Set

$$\Phi_\alpha(P) = \begin{cases} B_P(2 - |x|) & \text{if} \quad P \in U_\alpha, \psi_\alpha(P) = x \\ 0 & \text{otherwise;} \end{cases}$$

as in the proof of Proposition 1.3, $\Phi_\alpha(P)$ is smooth. The functions Φ_α have the properties:

i) For each $P \in M$, there is an α, with $\Phi_\alpha(P) = 1$.

ii) Each $P \in M$ has a neighbourhood on which all but a finite number of functions Φ_α vanish.

These, in fact, translate properties i) and ii) of the Theorem. By ii), the function $\sum_\alpha \Phi_\alpha(P) = \sum(P)$ can be defined, and is everywhere smooth; We set $\Psi_\alpha(P) = \Phi_\alpha(P)/\sum(P)$. Again using ii) we can define

$$g(P) = \sum_\alpha \delta_\alpha \Psi_\alpha(P);$$

since $\sum_\alpha \Psi_\alpha(P) = 1$, this is a weighted mean of numbers δ_α all less than $f(P)$, hence also is, and it is positive, as all $\delta_\alpha > 0$ and so is some $\Psi_\alpha(P)$. □

Complement 2.3. *We can find a countable set of pairs of disjoint coordinate discs (U_α, V_α) such that the $U_\alpha \times V_\alpha$ cover all $M \times M$ except the diagonal points (x, x).*

Proof. As above, we easily reduce this to a problem in Euclidean space, and there the disjoint pairs of $U(x, \sqrt{m}/i)$, (where ix has integer coordinates) will clearly do what we want. □

Definition 2.4. A set of nonnegative smooth functions ψ_α on M is called a *partition of unity* if the sets $U_\alpha = \{P\colon \psi_\alpha(P) > 0\}$ form a locally finite covering of M and $\sum_\alpha \Psi_\alpha(P) = 1$.

The functions Ψ_α above had this property, and, in addition, that the closure of the U_α were compact.

Our next investigation of smooth manifolds concerns connectedness.

Definition 2.5. A smooth map $\alpha\colon \mathbb{R} \to M$ is called a *path* in M. Two points P, Q in M are called *connected in* M if there is a path in M whose image contains P and Q.

Lemma 2.6. *Connectedness in M is an equivalence relation.*

Proof. By definition, the relation is symmetric. It is reflexive, since a constant map is a path. To prove transitivity, let α, β, be paths with images containing $(P, Q)(Q, R)$, and suppose without loss of generality $\alpha(-1) = P$, $\alpha(0) = Q$, $\beta(0) = Q$, and $\beta(1) = R$. Let $\varphi\colon U \to V$ be a C.N. of Q such that V is convex. Sinceα is continuous, for some $\varepsilon > 0$ and < 1, $|t| < \varepsilon \Rightarrow \alpha(t) \in U$. Similarly for β; let's suppose that ε will do for both. Now define by

$$\gamma(t) = \begin{cases} \alpha(t) & t < -\varepsilon \\ (1-\lambda)\alpha(t) + \lambda\beta(t) & -\varepsilon \leqq t \leqq \varepsilon \\ \beta(t) & t > \varepsilon \end{cases}$$

where the linear combination is taken in V and λ is a smoothing function which is 0 near $t = -\varepsilon$ and 1 near $t = \varepsilon$, e.g.,

$$\lambda(t) = B_P(t/\varepsilon - 1/2)$$

then γ is clearly smooth, and its image contains P and R. □

Lemma 2.7. *Each equivalence class is open in M.*

Proof. If $\varphi\colon U \to V$ is a C.N. of P such that V is convex, every point of U can be joined to P using the path corresponding to the straight line in V (suitably parametrised). □

Corollary 2.8. *Each equivalence class is closed in M.*

Proof. Each equivalence class is the complement of the union of the other equivalence classes. □

Lemma 2.9. *A subset of M is open and closed if and only if it is a union of equivalence classes.*

Proof. Sufficiency follows by Lemma 2.7 and Corollary. For necessity, observe that since $\mathbb{R}^1$ is connected, any path which meets an open and closed subset is contained in it, so such a subset is saturated for the equivalence relation. □

Definition 2.10. The equivalence classes are called the *components* of M. M is *connected* if it only has one component.

Lemma 2.9 shows that this is taking components in their usual sense. Comparing with Definition 2.5, we note that for smooth manifolds, connection and connection by smooth paths are equivalent. A component of M, being open, is an open submanifold; and M is the discrete union of all its components. Thus to study M up to diffeomorphism, it suffices to take the components separately; we shall frequently do this.

Proposition 2.11. *A connected smooth manifold M^m has a countable base of open sets.*

Proof. Since $\mathbb{R}^m$ has a countable base, it is sufficient to show that the set of neighbourhoods occurring in Theorem 2.1 is countable. Since M^m is connected, there is a path joining any two points, so between any two neighbourhoods $\varphi_\alpha^{-1}(U(0,1))$ there is a finite chain of such neighbourhoods, each overlapping the next. Now the sets U_α are locally finite: each point has a neighbourhood meeting only a finite number, so each compact set meets only a finite number. Thus each $\varphi_\alpha^{-1}(\bar{U}(0,1)$ and so each $\varphi_\alpha^{-1}(U(0,1))$ meets only a finite number of others. By induction, the number of sets $U(0,1)$ joined to a given one by a chain of length at most k is finite; hence their total number is at most countably infinite. □

Corollary 2.12. *A smooth manifold M^m is second countable if and only if the set of its components is finite or enumerable.*

Lemma 2.13. *Let Y be a metric space, X a closed subset. For any open neighbourhood U of X in Y, there is a positive continuous function f on X such that $x \in X, \varphi(x,y) < f(x) \Rightarrow y \in U$.*

Proof. Define $f(x) = \varphi(x, Y \setminus U)$: clearly $|f(x) - f(x')| \leqq \varphi(x,x')$, so f is continuous: it is nonzero and satisfies the condition. □

Corollary 2.14. *If X is a compact subset of the metric space Y, any open neighbourhood U of X in Y contains an ε-neighbourhood for some $\varepsilon > 0$.*

Proof. Take $\varepsilon = \inf f$, where f is given by the Lemma. □

Corollary 2.15. *If X is a metric space, U a neighbourhood of the diagonal $\Delta(X) \in X \times X$, there is a positive continuous function f_1 on X such that $\varphi(x,y) < f_1(x) \Rightarrow (x,y) \in U$.*

Proof. Take $Y = X \times X$ and φ_1 a product metric in the Lemma, and set $f_1(x) = (f(x,x))$. Since $\varphi(x,y) = \varphi_1((x,y),(x,y))$ the result follows. □

Corollary 2.16. *If X is a compact metric space, U a neighbourhood of $\Delta(X)$ in $X \times X$, then for some $\varepsilon > 0$, $\varphi(x,y) < \varepsilon \Rightarrow (x,y) \in U$.*

Proof. Take $\varepsilon = \inf f_1$, where f_1 is given by Corollary 2.15. □

Lemma 2.17. *If X is a compact subset of the metric space Y, and U an open neighbourhood of $X \times X$ in $Y \times Y$, then for some $\varepsilon > 0$, if V is the ε-neighbourhood of X in Y, U contains $V \times V$.*

Proof. Take $\varepsilon = \frac{1}{2}\varphi(X \times X, Y \times Y \setminus U)$, which exists since $Y \times Y$ is normal, $X \times X$ is compact. Then if $\varphi(v_1, X) < \varepsilon$, $\varphi(v_2, X) < \varepsilon$ we have $\varphi((v_1 \times v_2, X \times X) < 2\varepsilon = \varphi(X \times X, Y \times Y \setminus U)$, so $v_1 \times v_2$ does not lie in $Y \times Y \setminus U$. □

Corollary 2.18. *Let X be a compact subspace of the metric space Y, $f \colon Y \to Z$ be locally homeomorphic, and $f|X$ be 1 - 1. Then X has a neighbourhood U in Y such that $f|U$ is a homeomorphism.*

Proof. Let $D = \{(y_1, y_2) \colon y_1 \neq y_2, f(y_1) = f(y_2)\} \subset Y \times Y$. By hypothesis, D is disjoint from $X \times X$ (since $f|X$ is 1 - 1). Now the closure $\bar{D}$ is contained in the closed subset defined by $f(y_1) = f(y_2)$, so is contained in $D \cup \Delta(Y)$. But by hypothesis, f is a local homeomorphism, so each point (y, y) has a neighbourhood disjoint from D. Thus $\bar{D}$ is disjoint from $\Delta(Y)$, so D is closed. Now apply Lemma 2.17, taking $U = Y \times Y \setminus D$. We find V, so that $V \times V$ does not meet D. Hence $f|X$ be 1 - 1, so is a homeomorphism. □

Lemma 2.19. *Let V be locally compact, N Hausdorff. Then a proper 1 - 1 map $f \colon V \to N$ is a homeomorphism onto its image.*

Proof. Let $M = f(V)$. Since f is proper onto M it extends to a continuous map of the one-point compactifications

$$\bar{f} \colon V \cup \infty \to M \cup \infty.$$

$\bar{f}$ is a 1 - 1 map of a compact set, so a homeomorphism. Hence f is a homeomorphism. □

Complement 2.20. *If $f \colon V \to N$ is proper, then M is closed in N.*

Proof. Define $\bar{f} \colon V \cup \infty \to N \cup \infty$ for f. Then $\bar{f}$ is a homeomorphism into, with compact image. Since $M \cup \infty$ is closed in $N \cup \infty$, so is M in N. □

Theorem 2.21. *(Baire's theorem) Let M be a complete metric space. The intersection of a countable family of dense open subsets of M is dense.*

Proof. Let the given subsets be $\{U_i\}$, and let V be any nonempty open set. Then $V \cap U_1$is nonempty and open, and so contains a spherical neighbourhood $U(x_1, \varepsilon_1)$. Next, $U_2 \cap U(x_1, \varepsilon_{1/2})$ is nonempty and open, so contains a $U(x_2, \varepsilon_2)$. We can thus construct a decreasing sequence of neighbourhoods $U(x_i, \varepsilon_i)$ and clearly $\varepsilon_i \to 0$. Then $\{X_i\}$ is a Cauchy sequence, so has a limit point X, which lies in each $\bar{U}(X_i, \varepsilon_i)$ (since the later X_j do) and so in each U_i and in V. □

Complement 2.22. *If W is open in M, the theorem holds for W.*

Proof. We construct the neighbourhoods as above. The limit point X exists in M (which is complete), and hence by the argument above also in W. □

Chapter 3

Tangent Vectors

Throughout this chapter, M^n will be a smooth manifold.

Definition 3.1. A *tangent vector* at $P \in M$ is a derivation on $\mathcal{F}$ to $\mathbb{R}$. More precisely, it is a mapping $\lambda\colon \mathcal{F} \to \mathbb{R}$ which satisfies

i) If $a_1, a_2 \in \mathbb{R}$, $f_1, f_2 \in \mathcal{F}$, $\lambda(a_1 f_1 + a_2 f_2) = a_1\lambda(f_1) + a_2\lambda(f_2)$.

ii) If $f_1, f_2 \in \mathcal{F}$, $\lambda(f_1 f_2) = \lambda(f_1) f_2(P) + f_1(P)\lambda(f_2)$.

We shall discuss the structure of the set of all tangent vectors to M. Note that sums and real multiples of tangent vectors at P are also tangent vectors at P, thus these form a vector space.

Definition 3.2. The *tangent space* M_P to M at P is the vector space formed by all tangent vectors to M at P.

Let $\varphi\colon U \to V$ be a C.N. of P, and suppose without loss of generality $\varphi(P) = 0$. Let $x_1, \ldots, x_n$ be coordinates in $\mathbb{R}^n$. Then for each $f \in \mathcal{F}$, we have $f' = f \circ \varphi^{-1}$, a smooth function on V, so there are partial derivatives $d_i f = \frac{\partial f'}{\partial x_i}|_0$. We assert that d_i is a tangent vector at P: condition i) is clear, and ii) follows by the rule for differentiating a product. We shall prove that these form a basis for M_P; first, however, we need a lemma.

Lemma 3.3. *Let f be a smooth function on an open convex subset V of $\mathbb{R}^m$ containing 0, and let $f(0) = 0$. Then there exist further smooth functions $f_i (1 \leqq i \leqq m)$ on V such that $f(x) = \sum_1^m x_i f_i(x)$. Moreover, if f is a smooth function of additional parameters a_j, we may suppose that f_i also are.*

Proof.

$$\begin{aligned} f(x-0) = f(x) - f(0) &= \int_0^1 \frac{\partial f(tx)}{\partial t} dt \\ &= \int_0^1 \sum_1^m x_i \frac{\partial f}{\partial x_i}(tx) dt \\ &= \sum_1^m x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx) dt \end{aligned}$$

Hence we can take $f_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(tx)dt$. The last part also follows. □

Theorem 3.4. *The tangent vectors $d_1, \ldots, d_m$ form a basis for M_P.*

Proof. We first remark that a tangent vector is essentially local in nature: if $f = g$ in a neighbourhood U of P, and λ is a tangent vector at P, then $\lambda(f) = \lambda(g)$. For by Proposition 1.3, we can find a function Φ on M, equal to 1 in a neighbourhood of P, and zero outside U. Then $\Phi f = \Phi g$, and so $f - g = (f-g)(1-\Phi)$. Thus

$$\begin{aligned} \lambda(f) - \lambda(g) = \lambda(f-g) &= \lambda(f-g)(1-\Phi(P)) + (f(P) - g(P))\lambda(1-\Phi) \\ &= 0. \end{aligned}$$

Hence it is sufficient to consider only functions defined and smooth in U, where $\varphi \colon U \to V$ is a C.N. of P with V convex; it will be simpler to speak directly of functions on V.

For any smooth function f on V, by Lemma 3.3, we can put

$$f(X) = f(0) + \sum x_i f_i(x).$$

For any tangent vector λ at P, then,

$$\begin{aligned} \lambda(f) &= \lambda(f(0)) + \sum \lambda(x_i, f_i) \\ &= f(0)\lambda(1) + \sum \lambda(x_i) f_i(0) + \sum x_i(0)\lambda(f_i). \end{aligned}$$

But

$$\lambda(1) = \lambda(1 \cdot 1) = 1 \cdot \lambda(1) + \lambda(1) \cdot 1 = 2\lambda(1),$$

and so $\lambda(1) = 0$. Thus

$$\lambda(f) = \sum \lambda(x_i) f_i(0).$$

In particular

$$d_j(f) = \sum d_j(x_i f_i(0) = \sum \delta_{ij} f_i(0) = f_j(0).$$

So

$$\lambda(f) = \sum \lambda(x_i) d_i(f)$$

and as this is true for all f, $\lambda = \sum \lambda(x_i) d_i$. Hence the d_i span M_P. Since $d_i(x_j) = \delta_{ij}$, they are linearly independent. Hence they form a basis. □

We shall usually, by abuse of notation, write $\partial/\partial x_i$ for d_i. Now let $\varphi\colon M^m \to V^v$ be a smooth mapping, and let $\varphi(P) = Q$.

Definition 3.5. The *differential* of φ at P, $d\varphi\colon M_P \to V_Q$ is defined by:

$$d\varphi(x)(f) = X(f \circ \varphi) \quad \text{for} \quad X \in M_P, f \in \mathcal{F}_V.$$

Since f, φ are smooth, so is $f \circ \varphi$, so the right hand side is defined. Then $d\varphi(x)$ is a derivation since X is. Clearly, $d\varphi$ is a linear mapping of M_P to V_Q.

If $f \in \mathcal{F}_m, f\colon M^m \to \mathbb{R}$ is a smooth mapping, so that if $f(P) = a$, we have $df\colon M_P \to \mathbb{R}_a$. However, we may identify each $\mathbb{R}_a$ with $\mathbb{R}$ itself in a natural manner: if x is the parameter on $\mathbb{R}$, identify the vector $k\partial/\partial x$ with the number $x = k$. By change of parameter $y = \lambda x$, we have the same identification. [Similarly, we identify tangent spaces to $\mathbb{R}^n$ with $\mathbb{R}^n$ itself.] Thus for $f \in \mathcal{F}_m, P \in M$, we have $df\colon M_P \to \mathbb{R}$. Since df is linear, it is an element of the dual vector space, M_P^* to M_P. Now, if $x_1, \ldots, x_m$ are local coordinates at P, we have

$$dx_i(\partial/\partial x_j) = \partial x_i/\partial x_j = \delta_{ij}$$

so the dx_j form the basis of M_P^* dual to the basis of $(\partial/\partial x_i)$ of M_P.

This concludes the discussion of tangent vectors at a single point. We now wish to assemble together all tangent vectors: for this we need the idea of a fibre bundle. We refer the reader to Steenrod's book "The Topology of Fibre Bundles" for a fuller description; we shall recapitulate some definitions here for the sake of continuity of argument.

Definition 3.6. A map $\pi\colon T \to M$ is the projection of an *n-vector bundle* if M can be covered by open sets U_α such that

i) There are homeomorphisms

$$\varphi_\alpha\colon U_\alpha \times \mathbb{R}^n \to \pi^{-1}(U_\alpha)$$

such that $\pi\varphi_\alpha(m, x) = m$.

ii) For each pair (α, β) there is a continuous map

$$g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathbf{GL}_n(\mathbb{R})$$

such that for $m \in U_\alpha \cap U_\beta, X \in \mathbb{R}^n$,

$$\varphi_\beta(m, x) = \varphi_\alpha(m, g_{\alpha\beta}(m), X).$$

A map $\chi\colon M \to T$ is called a *cross-section* if $\pi \circ \chi = 1$. The bundle is *smooth* if the maps $g_{\alpha\beta}$ are smooth [$\mathbf{GL}_n(\mathbb{R})$ is an open submanifold of $\mathbb{R}^{n^2}$.] In this case T admits a natural structure as smooth $(m+n)$-manifold, such that the maps φ_α are diffeomorphisms on open submanifolds. For if we use these to define C.N.s, then we have differentiable transformations of coordinates on the intersections.

For a general *fibre bundle*, $\mathbf{GL}_n(\mathbb{R})$ is replaced by a general topological group G (we shall only make use of Lie groups) and $\mathbb{R}^n$ by a general topological space F (the *fibre*) on which G operates. The structure of the bundle is determined by the maps $g_{\alpha\beta}$; two bundles with the same $g_{\alpha\beta}$ but different fibres are called *associated*. If the $g_{\alpha\beta}$ all have images in a subgroup G' of G, we say that the group of the bundle *reduces* to G'.

Write $\Pi(M) = \cup\{M_P \colon P \in M\}$; the set of all tangent vectors to M. Define $\pi\colon \Pi(M) \to M$ by $\pi(M_P) = P$. Let $H_\alpha\colon U_\alpha \to V_\alpha$ be a set of local coordinate systems, with the U_α covering M, and for $P \in U_\alpha, v \in \mathbb{R}^m$, define $\varphi_\alpha(P, v)$ as the tangent vector at P determined by $\sum v_i\partial/\partial x_i$. Then $\varphi_\alpha\colon U_\alpha \times \mathbb{R}^m \to \pi^{-1}(U_\alpha)$ is a $1-1$ mapping for each α. On $U_\alpha \cap U_\beta$, denoting the two systems of coordinates by x^α, x^β; we have, by the usual transformation rule

$$\partial/\partial x_i^\beta = \partial x_j^\alpha/\partial x_i^\beta \partial/\partial x_j^\alpha,$$

so we define $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathbf{GL}_m(\mathbb{R})$ by

$$g_{\alpha\beta}(Q) = (\frac{\partial x_j^\alpha}{\partial x_i^\beta})_Q.$$

Then $g_{\alpha\beta}$ is a smooth mapping, and satisfies the condition above. To conclude that we have a vector bundle, it remains only to topologise $\Pi(M)$. But since the maps $g_{\alpha\beta}$ are smooth, we may as above take the φ_α (or rather their inverses) as C.N.s, and thus define on $\Pi(M)$ the structure of smooth manifold, which in particular gives it a topology, with the φ_α homeomorphisms.

Definition 3.7. $\pi(M)$ is the *tangent bundle* to M. Write $\Pi_0(M)$ for the zero cross-section, i.e., the set of zero tangent vectors. In general, a smooth cross-section of $\Pi(M)$ is called a *vector field* on M. Any bundle associated to $\Pi(M)$ via a linear representation of $\mathbf{GL}_n(\mathbb{R})$ is called a *tensor bundle* (and a points of it are tensors, whose type is determined by the representation). The bundle given by the adjoint representation is the *bundle of differential 1-forms* on M^m; its fibre over P is the dual space M_P^* to M_P. The bundle whose fibre over P is the set of all positive definite quadratic forms on M_P is called the *Riemann bundle*, and any cross-section of it a *Riemannian structure* on M.

For further discussion of such bundles, we refer the reader to Nomizu's book 'Lie Groups and Differential Geometry'. The above contains more than we shall need. We now prove the fundamental

Theorem 3.8. *Every smooth manifold M^m has a Riemannian structure.*

Proof. Let $\varphi_\alpha\colon U_\alpha \to U(0,3)$ be the C.N.s constructed in Theorem 2.1 and let Ψ_α be the partition of unity constructed in the Corollary. Now $U(0,3)$ has the standard Euclidean Riemannian structure: $\sum dx_i^2$. We write $ds^2 = \sum \Psi_\alpha \sum (dx_i \circ d\varphi_\alpha)^2$. As usual, since the U_α are locally finite, th sum is defined. Since a linear combination of positive definite quadratic forms is again positive definite, ds^2 is everywhere positive definite. Thus it defines a Riemannian structure on M^m. □

Now suppose a Riemannian structure chosen on M^m. This induces an inner product on each M_P, which we use to introduce notions of length of tangent vector, etc. We can modify the maps $\varphi_\alpha \colon U_\alpha \times \mathbb{R}^m \to \pi^{-1}(U_\alpha)$ so as to preserve the inner product on the fibres; simply apply the Gram-Schmidt orthogonalisation process. In fact, consider φ_α as a map $\varphi \colon \mathbb{R}^m \to \mathbb{R}^m$ depending on certain parameters. We modify φ by putting

$$\varphi'(e_i) = \sum_{j \leqq i} \lambda_{ij} \varphi(e_j)$$

where the λ_{ij} with $j < i$ are chosen to make the $\varphi'(e_i)$ orthogonal and $\lambda_{ii} > 0$ so as to make the $\varphi'(e_i)$ unit vectors. Then the λ_{ij} are also smooth functions of the parameters.

Thus if a Riemannian structure is chosen on M^m, we can always consider orthonormal bases in the fibres, so the group then reduces to the orthogonal group $\mathbf{O}(n)$. The converse: that a reduction to $\mathbf{O}(n)$ corresponds to a Riemannian structure, follows by reversing the argument. We observe that the choice of an inner product on M_P allows us to identify M_P with M_P^*. For a Riemannian manifold, we shall usually do this.

Definition 3.9. M^m is called *orientable* if the group of the tangent bundle is reducible to $\mathbf{GL}_n^+(\mathbb{R})$, *oriented* if the group is so reduced. Since the coordinate transformations were given by the matrices $(\partial x_j^\alpha / \partial x_i^\beta)$ the condition is that all the Jacobian determinants are positive. The bundle associated to the tangent bundle with fibre $\mathbf{GL}_m(\mathbb{R})/\mathbf{GL}_m^+(\mathbb{R}) = \mathbb{Z}_2$ is a double covering of M, called the *orientation covering*. Its projection on M, together with C.N.s of M, can be taken as C.N.s, so the orientation covering is a smooth manifold. By the definition, all the Jacobians occurring here are positive, so this manifold is orientable. If M is non-orientable, we can find a closed chain of C.N.s, each overlapping the next, such that the number of negative Jacobians is odd.

If M has a Riemannian structure, the same considerations of orientation apply, replacing $\mathbf{GL}_m(\mathbb{R}), \mathbf{GL}_m^+(\mathbb{R})$ by $\mathbf{O}_m, \mathbf{SO}_m$.

Chapter 4

Analysis

In this chapter, we list a number of standard results from analysis which we shall need later. Since a number of the proofs are long, we shall omit them, and give references for the less accessible results.

Theorem 4.1. *Inverse Function Theorem*

Let $f_1, \ldots, f_n$ be smooth functions defined in a neighbourhood of $O \in \mathbb{R}^n$ and suppose $|\frac{\partial f_i}{\partial f_j}| \neq 0$ at O. Then in some neighbourhood U of O, $f_1, \ldots, f_n$ define a diffeomorphism of U on an open subset of $\mathbb{R}^n$

Corollary 4.2. *Let M^m be a smooth manifold; $f_1, \ldots, f_n$ be smooth functions on M, $P \in M$. The f_i may be taken as coordinate functions for a C.N. of P if and only if the df_i form a basis for M_P^*.*

Proof. Let $\varphi\colon U \to \mathbb{R}^n$ be a C.N. of P. Then the $f_i \circ \varphi^{-1}$ are smooth functions on a neighbourhood of $\varphi(P) \in \mathbb{R}^n$; by the theorem, they define a diffeomorphism of some such neighbourhood if and only if the Jacobian $|\frac{\partial(f_i \circ \varphi^{-1})}{\partial x_j}| \neq 0$ at $\varphi(P)$. But the elements of this matrix are just the coefficients in the df_i of basis elements dx_j of M_P^*. □

Theorem 4.3. *Implicit Function Theorem*

Let $f_1, \ldots, f_r$ be smooth functions defined in a neighbourhood of $O \in \mathbb{R}^{r+s}$ and suppose the determinant formed by their partial derivatives with respect to $x_1, \ldots, x_r$ is nonzero at O. Then there are r smooth functions $g_1, \ldots, g_r$ defined in a neighbourhood of $O \in \mathbb{R}^s$ such that within some neighbourhood of $O \in \mathbb{R}^{r+s}$, a point satisfies $f_i(P) = 0$ $(1 \leqq i \leqq r)$ if and only if it satisfies

$$x_i = g_i(x_{r+1}, \ldots, x_{r+s}) \quad (1 \leqq i \leqq r).$$

Theorem 4.4. *Whitney's Extension Theorem*

Let f be a smooth function defined on the open set $x_1 > 0$ of $\mathbb{R}^n$, and suppose that f and all its partial derivatives extend to continuous functions on $\mathbb{R}^n_+$. Then there is a smooth function g on $\mathbb{R}^n$ which agrees with f in its range of definition.

Whitney's proof, which establishes results of much greater generality, can be found e.g., in his paper: "Analytic extensions of differentiable functions defined on closed sets", in the Transactions of American Mathematical Society 36 (1934) pp. 63-89.

We next consider Picard's existence theorem for differential equations. It is convenient to use the following terms. Let U be an open subset of $\mathbb{R}^n$, K be a compact subset of U.

Theorem 4.5. *Existence Theorem for Ordinary Differential Equations*
Given a system of equations $\frac{dx}{dt} = \varphi(x)$ *where* φ *is a smooth function on* U *to* $\mathbb{R}^n$*, then for some* $\varepsilon > 0$ *there exists a unique smooth function* $x = g(x_0, t)$ *on* $K \times E$ *to* U *(where* E *is the set:* $|t| < \varepsilon$*) satisfying the equation, and such that* $x_0 = g(x_0, 0)$*.*

We shall use this to develop the connection between vector fields on a smooth manifold M^m and 1-parameter groups of diffeomorphisms of M.

Definition 4.6. A family $\{\varphi_t \colon t \in \mathbb{R}\}$ of mappings of M into itself is called a *1-parameter group of diffeomorphisms* of M if

i) The mapping $\varphi \colon M \times \mathbb{R} \to M \times \mathbb{R}$ defined by $\varphi(m, t) = (\varphi_t(m), t)$ is a diffeomorphism

ii) For all $s, t \in \mathbb{R}$, $\varphi_s \varphi_t = \varphi_{s+t}$.

We observe that the first condition implies that each φ_t is in fact a diffeomorphism. Now suppose $\{\varphi_t\}$ does satisfy these conditions. Then we define a vector field X on M as follows. For $f \in \mathfrak{F}_M$, $P \in M$, we set

$$X_P(f) = \lim_{t \to 0} \frac{f(\varphi_t(P)) - f(P)}{t} = \frac{d}{dt} f(\varphi_t(P))|_{t=0}.$$

It is clear that X_P is a tangent vector to M at P. The fact that X_P varies smoothly with P, so that X is a vector field, now follows from i).

Our present aim is to obtain a partial converse to this result.

Theorem 4.7. *Let* M^m *be a smooth manifold,* X *a vector field on* M*,* U *an open set in* M *with compact closure* K*. Then we can find* $\varepsilon > 0$*, and for each* t *with* $|t| < \varepsilon$*, a map* φ_t *of* U *in* M*, such that*

i) The map $\varphi \colon U \times E \to M \times \mathbb{R}$ *(defined as above) is a diffeomorphism onto an open submanifold.*

ii) If $|s|$*,* $|t|$*, and* $|s+t|$ *are less than* ε*;* P *and* $\varphi_t(P)$ *are in* U*, then* $\varphi_s\varphi_t(P) = \varphi_{s+t}(P)$*.*

iii) For each $P \in U$*,* $f \in \mathfrak{F}_M, X_P(f) = \frac{d}{dt} f(\varphi_t(P))|_{t=0}$*.*

The map φ *is uniquely determined by these conditions.*

Proof. Cover K by a finite number of compact sets K_α, each contained in the interior of V_α, where $H_\alpha\colon V_\alpha \to U_\alpha$ is a C.N. We shall now interpret our conditions in U_α. First, however, note that if $f \in \mathcal{F}$,

$$\begin{aligned}\frac{d}{dt}f(\varphi_t(P)|_{t=s} &= \frac{d}{dt}f(\varphi_{s+t}(P)|_{t=0}\\ &= \frac{d}{dt}f(\varphi_t((\varphi_s(P)))|_{t=0}\\ &= X_{\varphi_s(P)}(f)(1).\end{aligned}$$

Now in U_α write $X = \sum_1^n f^i\partial/\partial x_i$, and consider the system

$$\frac{dx_i}{dt} = f^i(x).$$

We shall apply Theorem 4.5, taking U_α for U, and $H(K_\alpha)$ for K. Since X is smooth, the f^i are smooth, and the result does apply: we find ε_α, and a smooth function $X = g(X_0, t)$ for $X_0 \in K_\alpha, |t| < \varepsilon$, uniquely determined by the equation. We write $\varphi_t(x_0) = g(x_0, t)$ - or rather define φ_t in M by this relation in U_α. If $\varepsilon = \min\,\varepsilon_\alpha$, φ_t is now defined on the required range: the fact that the functions defined by different C.N.s agree on the intersection follows by the uniqueness, and the fact that the equations solved are simply derived from each other by change of variables.

We note that the functions $\varphi_{s+t}(P) \to g(x_0, s+t)$ satisfy the same equation, with initial value $g(x_0, s)$. By the uniqueness, $g(x_0, s+t) = g(g(x_0, s), t)$, i.e., $\varphi_{s+t}(x_0) = \varphi_t\varphi_s(x_0)$ so each φ_t is a diffeomorphism (over a smaller set than K, initially - but we could have enlarged K in the first place), and since φ is smooth, it too is a diffeomorphism. □

Corollary 4.8. *If M^m is compact, each vector field generates a 1-parameter group of diffeomorphisms of M.*

Proof. We can now take $K = U = M$ in the theorem, and find $\varphi\colon M \times E \to M \times \mathbb{R}$. But the definition of φ can be extended over the whole $\mathbb{R}$ using the functional equation $\varphi_s\varphi_t = \varphi_{s+t}$, since this is satisfied in $|t| < \varepsilon$. □

In general, a vector field on M is called *complete* if it generates a 1-parameter group of diffeomorphisms of M.

Corollary 4.9. *If X is complete, and Y agrees with X outside a compact subset of M, then Y is also complete.*

Proof. Outside a neighbourhood of such a subset, φ can be defined for $|t| < \varepsilon$, by hypothesis, since it can for X. But such a neighbourhood is compact, so inside it φ can also be defined for $|t| < \varepsilon$, by the Theorem. The condition follows as for the first Corollary. □

We observe that in $\mathbb{R}$ the constant vector field $\partial/\partial x$ is complete; indeed, we then have $\varphi_t = t + x$. More generally, in the product $M^m \times \mathbb{R}$, the field which we may call $\partial/\partial t$ which maps to zero on the first factor and to the standard field on the second is also complete; here we have

$$\varphi_t(x, s) = (x, s + t).$$

These results are our first justification of the use of the term tangent in tangent vectors, since we now see that such vectors correspond to displacement along the manifold.

Part I

Geometrical Foundations

Chapter 1

Geodesics

In this chapter, we shall suppose that M^m has a fixed Riemannian structure ds^2, expressed in local coordinates by $ds^2 = g_{ij}dx_i dx_j$, where g_{ij} is a positive definite quadratic form. Let $p\colon \mathbb{R}^1 \to M$ be a path (smooth map). We define the *length* and *energy* of p between two of its points by

$$l(p) = \int_a^b \frac{ds}{dt} dt$$
$$E(p) = (b-a)\int_a^b (ds/dt)^2 dt$$

where $(ds/dt)^2 = g_{ij}(dx_i/dt)(dx_j/dt)^2$, the derivatives being taken along the path. We define a distance function on M by

$$\varphi(P,Q) = \inf l(p)\colon p \quad \text{a path joining} \quad P \text{to} \quad Q.$$

Thus $\varphi(P,Q)$ is defined if and only if P, Q are in the same component of M; in fact we suppose M connected for the remainder of this chapter. We note that at a point, by changing coordinates, we can diagonalise $ds^2 = \sum_1^n a_i dx_i^2$, and it is then cleat that at this point, and so in a neighbourhood, its ratio to the Euclidean metric is bounded above and below by positive numbers. Hence the metric induces the given topology on M; we call it the *Riemannian metric*.

Definition 1.1. A *geodesic* is a smooth path $p\colon U \to M$ (U open in $\mathbb{R}^1$) giving an extremal value to the energy between any two of its points.

By Schwarz' inequality

$$\{l(p)\}^2 = \{\int_a^b \frac{ds}{dt}dt\}^2 \leqq \int_a^b dt \int_a^b (ds/dt)^2 dt$$
$$= (b-a)\int_a^b (ds/dt)^2 dt = E(p),$$

with equality if and only if ds/dt is constant, so that the curve is parametrised proportionately to arc length. If it is not, we clearly do not have an extremal value, as a first order change in parametrisation, making it more even, will give a first order decrease in E. Since any curve can be parametrised by arc length, the geodesic gives an extremal value also to the length of the path.

Proposition 1.2. *In local coordinates, geodesics are defined by the equations*

$$\frac{d^2x_i}{dt^2} + \Gamma^i_{jk}\frac{dx_j}{dt}\frac{dx_k}{dt} = 0$$

Proof. Euler's equation for the variational problem is

$$\frac{\partial f}{\partial x_i} = \frac{d}{dt}(\frac{\partial f}{\partial y_i}), \quad \text{where} \quad y_i = \frac{dx_i}{dt},$$

i.e.,

$$\begin{aligned}\frac{\partial g_{jk}}{\partial x_i}\frac{dx_j}{dt}\frac{dx_k}{dt} &= \frac{d}{dt}(2g_{ij}\frac{dx_j}{dt}) \\ &= 2g_{ij}\frac{d^2x_j}{dt^2} + 2\frac{\partial g_{ij}}{\partial x_k}\frac{dx_j}{dt}\frac{dx_k}{dt} \\ &= 2g_{ij}\frac{d^2x_j}{dt^2} + \frac{dx_j}{dt}\frac{dx_k}{dt}(\frac{\partial g_{ij}}{\partial x_k} + \frac{\partial g_{ik}}{\partial x_j})\end{aligned}$$

If g^{ij} is the inverse to g_{ij}, multiply by g^{lj} and reduce;

$$\frac{d^2x_l}{dt^2} + \frac{1}{2}g^{li}(\frac{\partial g_{ij}}{\partial x_k} + \frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{jk}}{\partial x_i})\frac{dx_j}{dt}\frac{dx_k}{dt} = 0$$

The coefficient of the last term is usually abbreviated to Γ^l_{jk}. □

Theorem 1.3. *Let $\varphi\colon V \to U$ be a C.N. in M, K a compact subset. Then there exists $\varepsilon > 0$ such that for $P \in K$, $v \in M_P$, and $|v| \leqq \varepsilon$, there is a unique geodesic $p(t)$ with $p(0) = P$, $\frac{d}{dt}p(t)|_{t=0} = v$; this is defined for $|t| < 2$, stays in U, and depends smoothly on p, v, t.*

Proof. We shall apply the Existence Theorem for Ordinary Differential Equations (4.5 of Part 0). Consider the system

$$\left.\begin{aligned} dx_i/dt &= y_i \\ dy_i/dt &= \Gamma^i_{jk}(x)y_jy_k \end{aligned}\right\}$$

where $x \in U$, $|y| < 3$ corresponds to the U of that theorem, and $x \in K|y| \leqq 2$ to its K. Then for some $\varepsilon > 0$, we find a unique solution $x = f(x_0, y_0, t)$, depending smoothly on all its arguments, and lying in U. Lifting to V by φ^{-1}, this gives a geodesic in M. To deduce the theorem, we need only change parameter by $t' = \frac{2}{\varepsilon}t$; this has the effect of multiplying the initial $\frac{d}{dt}p(t)$ by the inverse factor, and so altering the condition $|v| \leqq 2$ to $|V| \leqq \varepsilon$. □

Remark that the condition $v = \frac{d}{dt}p(t)|_{t=0}$ means that for $f \in \mathcal{F}$, $v(f) = \frac{d}{dt}p(t)|_{t=0}$. We shall refer to v as the *direction* of p at P.

Definition 1.4. Let $P \in M$, $v \in M_P$, and suppose that the geodesic with direction v at P can be defined for $|t| \leqq 1$. Then $\exp(P, v)$ is the point at $|t| = 1$ on the geodesic. exp is called the *exponential map*. We also write

$$\operatorname{Exp}(P, v) = (P, \exp(P, v)).$$

Note that the local existence and uniqueness of geodesics of Theorem 1.3 does not imply global existence, but does imply uniqueness in the whole range of existence (by applying the result to a sequence of points along the geodesic) given the initial point and direction.

Corollary 1.5. $\exp\colon V \to M$, $\operatorname{Exp}\colon V \to M \times M$ *are smooth maps defined on a neighbourhood* V *of* $\Pi_0(M)$ *in* $\Pi(M)$.

Proof. By Theorem 1.3, each point of $\Pi_0(M)$ has a neighbourhood on which they are defined. □

Proposition 1.6. *The Jacobian determinant of* Exp *is nonzero on* $\Pi_0(M)$.

Proof. For $P \in M$, let $\varphi\colon U \to \mathbb{R}^m$ be a C.N., and choose $x_1, \ldots, x_m$ as coordinates in M, $dx_1, \ldots, dx_m$ as coordinates in the fibres M_P; write the latter as $v_1, \ldots, v_m$, and write coordinates in $M \times M$ as $x_1, \ldots, x_m, z_1, \ldots, z_m$. Then we have $\operatorname{Exp}(x, v) = (x, z)$, so it remains to compute the partial derivatives of the z_i at 0. Now z is the point at $t = 1$ on the solution of the equation $\frac{dz}{dt} = y$ with initial condition $z = x$, $y = v_0$ i.e., at the point t_0 on the solution with initial condition $z = x$, $y = v_0/t_0 = v$. Hence $z = x + t_0 v +$ smaller terms (where t_0 is small, v fixed), and so to find $\frac{\partial z_i}{\partial v_j}$, set $(v_0)_i = t_0 \delta_{ij}$; then

$$\frac{\partial z_i}{\partial v_j} = \frac{\partial z_i(v_0)}{\partial t_0}\Big|_{t_0=0} = \delta_{ij}$$

This proves the result: for later reference note also

$$\frac{\partial z_i}{\partial x_j} = \delta_{ij} \quad \text{(clear).}$$

□

Corollary 1.7. $\Pi_0(M)$ *has a neighbourhood* V' *in* $\Pi(M)$ *on which* Exp *is defined, and is a local diffeomorphism.*

Proof. Follows from the Proposition and the Inverse Function Theorem (Part 0, 4.1). □

Corollary 1.8. *If* M *is compact,* $\Pi_0(M)$ *has a neighbourhood* V'' *in* $\Pi(M)$ *on which* Exp *is defined, and is a diffeomorphism.*

Proof. Follows from 1.7, using Corollary 2.18 of Part 0. □

However, the result of the last corollary can also be obtained, in a stronger form, without assumption of compactness.

Theorem 1.9. *There is a neighbourhood W of $\Delta(M)$ in $M \times M$ such that if $(x, y) \in W$, there is a unique geodesic from x to y of length $\varphi(x, y)$. Hence* Exp *defines a diffeomorphism of* $\mathrm{Exp}^{-1}(W)$ *onto W.*

Proof. For $P \in M$, by Corollary 1.7, let U be a neighbourhood of P such that Exp^{-1} defines a diffeomorphism of $U \times U$ on a neighbourhood of $\Pi_0(U)_j$ and let $\varphi\colon U \to \mathbb{R}^n$ be a C.N. of P. Then if U_1 is a sufficiently small neighbourhood of P, each pair of points in U_1 is joined by a unique geodesic lying in U, and each geodesic going outside U is longer. We say that it is obvious that this geodesic gives a minimum length for curves in U joining the two points, by comparison with the Euclidean problem (in the technical language of Calculus of Variations, since the metric is positive definite, the problem is regular, and we have constructed a semi-field of extremals, passing through a point and covering a neighbourhood). Hence it gives the global minimum, which we defined as the distance $\varphi(x, y)$. Thus Exp^{-1} is a diffeomorphism on $U_1 \times U_1$: we take W as the union of such neighbourhoods. □

We recall that a metric space is complete if each fundamental sequence of points converges to a limit point, or equivalently, if each bounded closed subset is compact. With this concept, we can give the global forms of the above theorems.

Theorem 1.10. *M is complete if and only if geodesics may be indefinitely produced, i.e., if* exp *and* Exp *are definable on $\Pi(M)$. Any two points in a complete manifold may be joined by geodesics: the length of at least one such is the distance between them.*

Proof. Suppose first M is complete, and $p(t)$ a geodesic which exists only for $t < k$. Then its points form a fundamental sequence: since M is complete, these have a limit point P. But by Theorem 1.3, P has a compact neighbourhood K such that any geodesic within k may be produced a distance ε. This gives a contradiction.

Now suppose exp globally definable, but that there are pairs of points (P, Q) not joined by a geodesic of length $\varphi(P, Q)$. Let r be the greatest lower bound of the distance of such points Q from P (by Theorem 1.9, $r > 0$), let $K_1 = \{v \in M_p\colon |v| \leqq r\}$, and let $K = \exp(K_1)$. Then K_1 is compact, hence so is K, by definition of r, K contains all points at distance less than r from P. Choose $2\varepsilon < r$ as the number ε in Theorem 1.3, and choose Q such that $\varphi(P, Q) = r_0 < r + \varepsilon$, but P and Q are not joined by a geodesic of length $\varphi(P, Q)$. Now let P_i be a smooth path from P to Q of length at most $r_0 + 1/i$, and let R_i be the point on it at distance $r - \varepsilon$ from P. The R_i lie in the compact set K; let R be a cluster point. Then

$$\varphi(P, R) \leqq \limsup \varphi(P, R_i) = r - \varepsilon,$$
$$\varphi(R, Q) \leqq \limsup \varphi(R_i, Q) = r_0 - r + \varepsilon$$

so by the triangle inequality we have

$$\varphi(P,R) = r - \varepsilon, \quad \varphi(R,Q) = r_0 - r + \varepsilon.$$

By the definition of r, ε; P can be joined to R by a geodesic of length $r - \varepsilon$; R to Q by on of length $r_0 - r + \varepsilon$. If these met at an angle at Q, by cutting a corner, we could find a shorter path; a contradiction. Hence they have the same direction at Q, so by the uniqueness theorem form part of the same geodesic. Thus P is joined to Q by a geodesic of length $\varphi(R,Q)$: a contradiction.

Finally, suppose $\exp M_P = M$. Then a bounded set lies within a finite distance from P, so is contained in the image of a closed and bounded, hence compact, subset of M_P. But the image of this set is also compact, so the result follows. □

Theorem 1.11. *Any connected manifold has a Riemannian metric in which it is complete.*

Proof. We make a slight refinement of the proof of Theorem 3.8 in Part 0, asserting the existence of Riemannian structures. Let $\varphi_\alpha \colon U_\alpha \to U(0,3)$ be the C.N.s constructed in Theorem 2.1, Part 0, and define $\Phi_\alpha \in \mathfrak{F}_i$ by

$$\Phi_\alpha(P) = \begin{cases} B_P(2\frac{1}{2} - |x|) & \text{if} \quad P \in U_\alpha, \varphi_\alpha(P) = x \\ 0 & \text{if} \quad P \notin U_\alpha. \end{cases}$$

Then write $ds^2 = \sum \Phi_\alpha(\sum dx_i^2) \circ \varphi_\alpha$. As in the earlier proof, we see that this is a metric. In $\varphi_\alpha^{-1}(U(0,1\frac{1}{2}))$, it dominates the Euclidean metric, so the set of points at distance $\leqq 1/3$ from $\varphi_\alpha^{-1}(\bar{U}(0,1))$ is a closed subset of $\varphi_\alpha^{-1}(\bar{U}(0,2))$, so is compact. As in Theorem 1.10, it follows that all geodesics from a point of $\varphi_\alpha^{-1}(\bar{U}(0,1))$, and hence from any point of M, may be produced a distance at least $1/3$. Thus they can all be produced indefinitely. □

Chapter 2

Submanifolds and Tubular Neighbourhoods

Definition 2.1. A subset M^m of a smooth manifold N^n is a *submanifold* (of dimension m, codimension $n-m$) if for each point $P \in M$, there is a C.N. $\varphi\colon U \to \mathbb{R}^n$ of P in N such that $U \cap M = \varphi^{-1}(\mathbb{R}^m)$.

Note that by Part 0, Corollary 4.2 this is equivalent to the requirement that in a neighbourhood of each point of M, M is defined by the vanishing of $(n-m)$ functions with linearly independent differentials. For in the case above, M is defined by the vanishing of the last $(n-m)$ coordinate functions; while by that corollary, any set of functions with linearly independent differentials can be take as functions of a C.N. If M is a closed subset of N, we call it a *closed submanifold*.

With this definition, M^m has a natural structure of smooth m-manifold, given by the restriction to M of the functions of $\mathcal{F}_N$; the existence of C.N.s for M follows immediately from the definition. We call this the *induced structure* on M.

Definition 2.2. A map $f\colon V \to N$ between two smooth manifolds will be called an *imbedding* if $f(V)$ is a submanifold M of N, and f induces a diffeomorphism of V on M, where M has the induced structure.

Lemma 2.3. *If a smooth map $f\colon V^v \to N^n$ is an imbedding then for each $Q \in V$, if $f(Q) = P$, $df\colon V_Q \to N_P$ has rank v.*

Proof. We know f is an imbedding. Choose a C.N. at P as above, and let $x_1, \ldots, x_n$ be the coordinate functions on N. By definition of the induced structure, $x_1 \circ f, \ldots, x_n \circ f$ define a C.N. of Q in V say $y_i = x_i \circ f$. But then $df(\partial/\partial y_i) = \partial/\partial x_i$ and so df has rank v at Q. □

Definition 2.4. A map $f\colon V^v \to N^n$ between two smooth manifolds is called an *immersion* if f is smooth, and for all $Q \in V$, writing $f(Q) = P$, then $df\colon V_Q \to N_P$ has rank v.

Thus lemma 2.3 state that an imbedding is always an immersion. The converse is of course false (the 'figure of 8' curve in the plane shows that), bur we can prove a partial converse, which is the first step in constructing imbeddings - one of our main objects.

Lemma 2.5. *An immersion is an imbedding if and only if it is a homeomorphism into.*

Proof. Let $f\colon V^v \to N^n$ be an immersion which is a homeomorphism onto its image M. Let $Q \in V$, $f(Q) = P$, and choose a C.N. $\varphi\colon U \to \mathbb{R}^n$ of P in N such that $df^*(x_1), \dots, df^*(x_v)$ form a basis for V_Q - this is possible since f is an immersion. Write $y_i = x_i \circ f$: then since $dy_1, \dots, dy_v$ form a basis for V_Q by Part 0, Corollary 4.2, $y_1, \dots, y_v$ may be taken as coordinates in a neighbourhood of Q. Since the other y_i are smooth functions, by the definition of smooth manifold we can write

$$y_i = g_i(y_1, \dots, y_v) \qquad v < i \leqq n$$

in a neighbourhood of Q in V. Since f is a homeomorphism into, we have $x_i = g_i(x_1, \dots, x_v)$ in a neighbourhood of P in M. Thus M is locally defined by vanishing of the $n - v$ smooth functions

$$x_i - g_i(x_1, \dots, x_v)$$

which clearly have linearly independent differentials. So M is a submanifold, and it is now clear that f defines a diffeomorphism of V on M. □

Corollary 2.6. *An immersion of a compact manifold is an imbedding if and only if it is 1-1.*

Proof. For a 1-1 continuous map of a compact space is a homeomorphism. □

Corollary 2.7. *An immersion is an imbedding if and only if it is 1-1 and a proper map onto its image.*

Proof. For an imbedding is clearly 1-1 and proper onto its image, and if f is 1-1 and proper onto its image, then by Part 0, Lemma 2.19 it is a homeomorphism into, and by the Lemma, it is then an imbedding. □

Corollary 2.8. *An immersion is an imbedding as a closed submanifold if and only if it is 1-1 and proper.*

We now return to our consideration of a submanifold M^m of a manifold N^n. If $P \in M$ the inclusion $i\colon M \to N$ induces $d_i\colon M_P \to N_P$ of rank m, hence the adjoint map $d_i^*\colon N_P^* \to M_P^*$ also has rank m, and its kernel has rank $(n - m)$.

Definition 2.9. The kernel of $d_i^*\colon N_P^* \to M_P^*$ is called the *normal space* to M in N at P. The union of the normal space is the *normal bundle* $\mathbb{N}(N/M)$ of M in N.

We must check that the normal bundle is indeed a vector bundle over M. Let $\varphi\colon U \to \mathbb{R}^n$ be a C.N. of P in N with $U \cap M = \varphi^{-1}(\mathbb{R}^m)$; then in $U \cap M$ we may take $dx_{m+1}, \ldots, dx_n$ as a basis for the normal space. These give the local product maps φ_α required of a fibre bundle; as with the tangent bundle, the maps $g_{\alpha\beta}$ come from Jacobians on change of coordinates.

We usually suppose a Riemannian structure chosen on N, which also induces one on M. The distinction between N_P^* and N_P disappears, and in this case we can regard $\mathbb{N}(N/M)$ as a sub-bundle of the restriction $\Pi(N)|M$ of $\Pi(N)$ to M. We refer the reader again to Steenrod for definitions concerning bundles: the Whitney sum of two vector bundles over M may be roughly described by taking the direct sum of the fibres over each point.

Proposition 2.10. $\Pi(N)|M$ *is the Whitney sum of* $\mathbb{N}(N/M)$ *and* $\Pi(M)$*,*

Proof. Since all the above bundles are defined, and the latter two are sub-bundles of the first, it is sufficient to verify that at each point the fibre of the first is the direct sum of the latter two. Since we have a positive definite inner product, it will be sufficient to verify that the fibre V_P of $\mathbb{N}(N/M)$ over P is the orthogonal complement of the fibre M_P of $\Pi(M)$ in the fibre N_P of $\Pi(N)$, or , that it is the annihilator of M_P in N_P^*. But since d_i^* is adjoint to d_i, the kernel of d_i^* is certainly the annihilator of the image of d_i. □

We now apply the results of Chapter 1.

Proposition 2.11. *The Jacobian of* $\exp\colon \mathbb{N}(N/M) \to N$ *on* $\Pi_0(M)$ *is nonzero.*

Proof. Let $P \in M$, and let $\varphi\colon U \to \mathbb{R}^n$ be a C.N. of P in N such that $U \cap M = \varphi^{-1}(\mathbb{R}^m)$. Then if $x_1, \ldots, x_n$ are coordinates in $\mathbb{R}^n$, we can take as local coordinates in $\mathbb{N}(N/M)$ $x_1, \ldots, x_m$ (coordinates in M) and $v_{m+1}, \ldots, v_n$ (coordinates in the fibre) where $v_i = dx_i$. Now refer back to Lemma 2.3, where we showed that if $\exp(x, v) = z$, then $\frac{\partial z_i}{\partial x_j} = \frac{\partial z_i}{\partial v_j} = \delta_{ij}$ so that with respect to our coordinates, the Jacobian matrix is the unit matrix, so its determinant is nonzero. □

Corollary 2.12. $\exp\colon \mathbb{N}(N/M) \to N$ *is a local diffeomorphism at* $\Pi_0(M)$*.*

Proof. This follows from the Inverse Function Theorem (Part 0, Theorem 4.1). □

Corollary 2.13. *If M is compact,* $\Pi_0(M)$ *has a neighbourhood in* $\mathbb{N}(N/M)$ *on which* exp *is a diffeomorphism to a neighbourhood of M in N.*

Proof. Use the above corollary and Part 0, Corollary 2.18. □

In fact, we can both strengthen the last corollary, and remove the assumption of compactness, so will now do so.

Theorem 2.14. *M has a neighbourhood U in N such that each point P of U is joined by a unique geodesic of length $\varphi(P, M)$; this meets M orthogonally. Thus* $\exp^{-1}$ *defines a diffeomorphism of U on a neighbourhood of* $\Pi_0(M)$ *in* $\mathbb{N}(N/M)$*.*

Proof. Let $Q \in M$, and let $U_1 \subset U_0$ be neighbourhoods of Q as in the proof of Theorem 1.9: any two points in U_1 are joined by a unique geodesic of minimal length, and this lies in U_0. We may clearly also suppose that any path joining a point of U_1 to a point outside U_0 is longer than the diameter of U_1 (simply take U_1 smaller). Then for $P \in U_0$ the closest point to P in M lies in $U_0 \cap M$ (such a point exists by local compactness of M, if we assume, say, $\bar{U}_0$ compact - the minimising point cannot lie outside U_0). If U_2 stands in the relation to U_1 that U_1 does to U_0, then for $P \in U_2$, the closest point to P in M lies in $U_1 \cap M$, so is joined to P by a unique shortest geodesic, lying in U_0. This, then, is the shortest curve joining P to a point of M; we say it meets M orthogonally. For if not, by a small modification near where it meets M, we could make it shorter (take a path orthogonal to M, and smooth off the corner). If we take U as the union of the sets U_2, the first part of the theorem is proved. Taking $\exp^{-1}$ to be defined by the shortest geodesic, this, with Corollary 2.12, proves the second part. □

With this preparation, we are ready for the main results of this chapter, which give a preliminary description of the way in which a submanifold lies in a manifold by describing the structure of a neighbourhood of the submanifold. With the extra precision which will be given in Chapter 4, this constitutes one of our main tools for getting at the structure of manifolds.

N^n is still a manifold, with a Riemannian structure. M^m is a submanifold, with normal bundle $\mathbb{N}(N/M)$ - this has group $\mathbf{O}_{n-m}$. Let us write B for the associated disc bundle: precisely, B consists of vectors of $\mathbb{N}(N/M)$ of at most unit length.

Definition 2.15. A *tubular neighbourhood* of M in N is an imbedding $\psi\colon B \to N$ (as submanifold with boundary, see definition 3.3 for exact definition): extending the diffeomorphism of $\Pi_0(M)$ on M induced by projection.

As with C.N.s, the actual neighbourhood $\psi(B)$ is the more geometrical concept; but the mapping ψ is more convenient to work with. The above definition appears to involve the Riemannian structure; however, if we extend it by letting B be any $(n-m)$-disc bundle over M, we shall see in Chapter 4 that this gives no extra generality; in fact we prove there a theorem of uniqueness for tubular neighbourhoods. Here, we only obtain existence.

Theorem 2.16. *There exist a tubular neighbourhood of M in N.*

Proof. Let W be a neighbourhood of $\Pi_0(M)$ in $\mathbb{N}(N/M)$ mapped diffeomorphically by exp: its existence is guaranteed by Theorem 2.14. Using Part 0, Lemma 2.13, let f be a positive continuous function on M such that vectors in $(N/M)_P$, of length less than $f(P)$, are contained in W. By Part 0, Corollary 2.2, we can find a positive smooth function g on M such that $0 < g(P) < f(P)$ for all $P \in M$. We now define a diffeomorphism ψ. For each $P \in M$, $v \in (N/M)_P$, set

$$\psi(P, V) = \exp(P, g(P)v))$$

Multiplication by $g(P)$ in the fibre is possible since $g(P) \neq 0$, and we have $|v| \leqq 1 \Rightarrow |g(P)v| \leqq g(P) < f(P) \Rightarrow (P, g(P)v) \in W$. $\square$

Chapter 3

Boundaries

We now extend the notion of manifold by considering manifolds with boundary. In the sequel these will play as much part as manifolds; we have merely deferred the definition till this point to help concentrate ideas.

Definition 3.1. N^n is a smooth *manifold with boundary* (or bounded manifold) if it satisfies all the defining conditions of a smooth manifold, with the exception that we allow C.N.s to map onto open sets in $\mathbb{R}^n_+$ (as well as $\mathbb{R}^n$).

The image of points on $x_1 = 0$ are called *boundary points* of N; it is cleat that this property is preserved on change of C.N. Their union is the *boundary* of N, which we always denote by ∂N. We write $\mathring{N} = N \setminus \partial N$, the 'interior' of N. By defining this as an open submanifold, it may be considered as a manifold.

There are various corresponding extensions or the notion of submanifold.

Definition 3.2. A subset M of a manifold with boundary N is a *submanifold* if it satisfies the same conditions as when N is not bounded, except that the C.N. φ may map U to $\mathbb{R}^n$ or $\mathbb{R}^n_+$, and if $\bar{M} \cap \partial N = M \cap \partial N$.

Thus in a neighbourhood of a point of M, the pair (N, M) is locally like $(\mathbb{R}^n, \mathbb{R}^m)$ or $(\mathbb{R}^n_+, \mathbb{R}^m_+)$. Geometrically, we can say that M meets ∂N transversely (for precise definition of this, see Part II). M has an induced structure of manifold with boundary, just as above, and we observe that $\partial M = M \cap \partial N$. In a particular case, ∂M is empty, and M disjoint from ∂N; but then M is a submanifold of $\mathring{N}$.

Definition 3.3. If N^n is a manifold (without boundary), we define M^m to be a *submanifold with boundary* of N^n, if M^m satisfies the defining conditions for a submanifold, weakened to allow $U \cap M = \varphi^{-1}(\mathbb{R}^m_+)$ as an alternative possibly to $U \cap M = \varphi^{-1}(\mathbb{R}^m)$.

In this case, in a neighbourhood of a point of M, the pair (N, M) is locally like $(\mathbb{R}^n, \mathbb{R}^m)$ or $(\mathbb{R}^n, \mathbb{R}^m_+)$. Again, M has the induced structure of manifold with boundary.

To the new kinds of submanifold correspond new kinds of imbedding. No changes need to be made in Definition 2.2; to distinguish cases we speak of imbedding V as a submanifold, or, as a submanifold with boundary.

We have still not defined sufficiently many types of manifold, and must next discuss corners. For example, the unit interval I is a manifold with boundary, but the product $I \times I$ is a square, so has corners, and is a new kind of object.

Definition 3.4. N^n is a *submanifold with corner* if it satisfies the defining conditions for a smooth manifold, except that C.N.s may map into open sets in any of $\mathbb{R}^n$, $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$.

Points corresponding to $x_1 = 0$ (in the second case) or to $x_1x_2 = 0$ (in the third) form the *boundary* ∂N; topologically (as opposed to differentiably), N is a manifold with boundary, and ∂N the boundary. Points corresponding to $x_1 = x_2 = 0$ (in the third case) form the *corner*, $\angle N$, which is a smooth manifold of dimension $n - 2$.

Now if M_1,M_2 are manifolds with boundary, products of C.N.s of M_1,M_2 give C.N.s in $M_1 \times M_2$ which (up to a permutation of coordinates) are appropriate for a manifold N with corner. We observe that $\partial(M_1 \times M_2) = \partial M_1 \times M_2 \cup M_1 \times \partial M_2$ and $\angle(M_1 \times M_2) = \partial M_1 \times \partial M_2$. In this, as most other important cases, $\angle N^2$ separates ∂N into two parts; of course this is always true locally.

We only introduce on more kind of submanifold, as we are not rally interested in corners, except in so far as they occur naturally.

Definition 3.5. M^m is a *submanifold with boundary* of the manifold with boundaryN^n if $\bar{M} \cap \partial N = M \cap \partial N$ and at each point of M a C.N. may be found mapping the pair on an open set in one of $(\mathbb{R}^n, \mathbb{R}^m)$, $(\mathbb{R}^n, \mathbb{R}^m_+)$, $(\mathbb{R}^n_+, \mathbb{R}^m_+)$, $(\mathbb{R}^n_+, \mathbb{R}^m_{++})$.

Such an M has an induced structure of manifold with corner, and $\angle M$ separates ∂M into two parts, one $\partial M \cap \mathring{N}$ and the closure of the other $\partial M \cap \partial N = M \cap \partial N$. We now give generalisations of the notion of tubular neighbourhood.

Let M be a manifold with boundary, $\pi\colon B \to M$ the projection of a disc bundle, Σ the boundary Sphere-bundle of B, and $C = \pi^{-1}(\partial M)$. It is then clear that B has the structure of a smooth manifold with corner, and $\angle B = \Sigma \cap C$ separates ∂B into two parts, with closures Σ and C. (If M has no boundary, C is empty, and B a manifold with boundary; this was already assumed in Definition 2.15).

Now suppose N^n a manifold with boundary, M^m a submanifold, and B an $(n - m)$-disc bundle over M.

Definition 3.6. A *tubular neighbourhood* of M in N is an imbedding $\psi\colon B \to N$ as submanifold with boundary, extending the diffeomorphism of the zero cross-section on M induced by projection.

It is easy to see that $\psi(c) = \partial N \cap \psi(B)$ in this case. Of course, such imbeddings may not exist for every disc-bundle B, or indeed for any at all: we will show, however, that for some B they do.

Definition 3.7. A *tubular neighbourhood* of ∂N in N is an imbedding $\psi\colon \partial N \times I \to N$ as submanifold with boundary, extending the projection of $\partial N \times 0$ on ∂N.

We define this separately, since we do not call ∂N a submanifold of N. This completes our list of definitions; we now survey how the results of the two preceding chapters extend to the boundaries. Let N be a smooth manifold with boundary. Then N has a Riemannian metric - the proof is the same as before. The discussion of geodesics at non-boundary points is also the same as before. At boundary points P, we must distinguish between *inward-* and *outward-pointing* tangent vectors; in terns of a C.N. of P, these are vectors $\Sigma\lambda_i\partial/\partial x_i$ with $\lambda_1 > 0$ resp. $\lambda_1 < 0$. If $\lambda_1 = 0$, we call the vector tangent to the boundary; indeed, if $i\colon \partial N \to N$ is the inclusion map, such vectors form the image of d_i, so do come from tangent vectors of ∂N. It is now clear, from the differential equations, that local geodesics can be constructed for all inward-pointing tangent vectors and for no outward-pointing ones. Is is not determinate in general what happens to those tangent to the boundary; as examples, the reader may consider D^2 and the closure of $\mathbb{R}^2 \setminus D^2$, each with the usual metric. The results of Chapter 1, up to and including Proposition 1.6 now follow, in suitably modified forms (the remainder are mostly false in general).

Proposition 3.8. *There exists a tubular neighbourhood of ∂N in N.*

Proof. We can identify $\partial N \times I$ with the set of inward-pointing normal vectors to ∂N, of length at most 1 (including those of zero length), for as there is only one normal direction at a point of ∂N, a normal vector there is determined by its length. The proof of Proposition 2.11 and Theorem 2.14 now carries over to this case. □

This proposition enables us in most cases, when discussing manifolds with boundary, to avoid special difficulties arising at the boundary. Our first illustration of this is with geodesics.

Definition 3.9. A Riemannian metric on N is *adapted to the boundary* if ∂N is totally geodesic, i.e., if construction in N of geodesics for vectors tangent to ∂N is locally possible, and if such geodesics are completely contained in ∂N.

Lemma 3.10. *Let M^m be closed, with a Riemannian metric. Then the product metric for $N = M \times \mathbb{R}^1_+$ is adapted to the boundary.*

Proof. Let $x_1, \ldots, x_m$ be local coordinates in M, and x_0 the coordinate in $\mathbb{R}^1_+$. Then for the metric g_{ij} we have $g_{0j} = \delta_{0j}$. Hence one of the defining equations for geodesics is simply $d^2x_0/dt^2 = 0$. Thus if initially $x_0 = dx_0/dt = 0$, we have $x_0 = 0$ all along the geodesic, which thus stays in ∂N - as indeed one would expect. □

Proposition 3.11. *Every manifold with boundary has a Riemannian metric adapted to the boundary.*

Proof. By Proposition 3.8, if N is the manifold, ∂N has a tubular neighbourhood $\psi\colon \partial N \times I \to N$. Let φ be a metric on N, φ' the product of some metric on ∂N with the standard metric of I. We define a metric φ'' by

$$\varphi'' = \begin{cases} \varphi & \text{outside the image of } \psi \\ \varphi' + (\varphi - \varphi')B_P(3t-1) & \text{at } \psi(P,t). \end{cases}$$

The latter agrees with φ in a neighbourhood of $t = 1$, so is smooth everywhere; it is a Riemannian structure, as a positive linear combination of positive form is another, and it agrees with φ' near $t = 0$, so by Lemma 3.10, it is adapted to ∂N. □

Using a metric adapted to the boundary, we could go on to find analogues of all the results in Chapter 1 except Theorem 1.10. We are more interested in generalising the results of Chapter 2. First note that a submanifold M of N meets ∂N orthogonally if the normal vectors to M and ∂N at each point of ∂M are perpendicular.

Lemma 3.12. *Let N be a manifold with boundary, M a submanifold. Then N has a Riemannian metric in which M meets ∂N orthogonally.*

Proof. We construct a metric just as in Part 0, Theorem 3.8; the only point to watch is that M meets ∂N orthogonally in each of the partial metrics to be fitted together. But since M is a submanifold, at a point of ∂M, there is a coordinate map of an open set of (N, M) to $(\mathbb{R}^n_+, \mathbb{R}^m_+)$, end the Euclidean metric will do. Now when we fit these together, M continues to meet ∂N orthogonally. □

Corollary 3.13. *N has a metric adapted to the boundary in which M meets ∂N orthogonally.*

Proof. We take the metric of Lemma 3.12, and construct a corresponding tubular neighbourhood of ∂N in N. Since for $P \in \partial M$, a vector at P normal to ∂N is tangent to M, it is a 'generator' of such a tube. Hence using this tubular neighbourhood in Proposition 3.11, M continues orthogonal to ∂N in the metric there constructed. □

Theorem 3.14. *If N is a manifold with boundary, M a submanifold, then there exists a tubular neighbourhood of M in N.*

Proof. The argument of Proposition 2.11 and Theorems 2.14 and 2.16 can now be carried through in this case: to avoid overloading this chapter, we shall leave the details to the reader. □

We shall need one further theorem involving tubular neighbourhoods and boundaries. We retain the hypothesis of Theorem 3.14.

Theorem 3.15. *There is a tubular neighbourhood $\psi\colon \partial N \times I \to N$ of ∂N in N such that $\psi|\partial M \times I$ is a tubular neighbourhood of ∂M in M.*

Proof. Let $\varphi\colon B \to N$ be a tubular neighbourhood of M in N (with notations as above). Give M a Riemannian structure, and B the product structure. As B is locally a product, we can do this locally, and as the group of the bundle B is the orthogonal group, which preserves the standard Riemannian structure in the fibre, these local structures agree on their intersections, and define a global structure.

Now as in Proposition 3.11, we modify the Riemannian structure on N so as to agree with the above structure on B in a neighbourhood of M (using the bump function to smooth off). Then construct a tubular neighbourhood ψ for ∂N as in Proposition 3.8. We assert ψ has the required property; indeed, since in a neighbourhood of M the metric is the product constructed above, geodesics tangent to M are contained in M, as in Lemma 3.10. □

Our tubular neighbourhoods give a global form to one's vague idea that a submanifold is imbedded nicely in a manifold, in that they describe the topology of a whole neighbourhood of the submanifold. We wish to obtain also uniqueness theorems for tubular neighbourhoods; for this we need some rather different methods.

Chapter 4

Diffeotopy Extension Theorems

Let M^m, N^n be smooth manifolds, possibly with boundary.

Definition 4.1. A *weak diffeotopy* of M in N is an imbedding (possibly as submanifold with boundary)

$$h\colon M \times \mathbb{R} \to N \times \mathbb{R}$$

which is *level-reserving*, i.e., we can write

$$h(m,t) = (h_t(m),t)) \qquad m \in M, t \in \mathbb{R}.$$

It follows that each h_t is also an imbedding. h is called *normalised* if $h_t = h_0$ when $t \leqq 0$, and $h_t = h_1$ when $t \geqq 1$, and is then also called a weak diffeotopy between h_0 and h_1.

A *diffeotopy* of N is a diffeomorphism k of $N \times \mathbb{R}$ which is level-preserving, thus in particular it is a weak diffeotopy of N in N. It is called *normalised* if $k_t = 1$ when $t \leqq 0$, and $k_t = k_1$ when $t \geqq 1$.

The diffeotopy k of N *covers* the weak diffeotopy h of M in N if

$$k_t(h_0 m) = (h_t(m) \qquad \text{for } m \in M, t \in \mathbb{R}.$$

A weak diffeotopy covered by a diffeotopy of N is called a *strong diffeotopy*.

It is desirable to prove that weak diffeotopies are strong, for the following reason. It frequently happens that we are able to construct a weak diffeotopy - for example, if m is small compared to n (see next part), between imbeddings. If the diffeotopy is strong, there is a diffeomorphism (k_1) of N carrying one imbedding to another, so that up to diffeomorphism the imbeddings are the same. The diffeotopy extension theorem asserts that under certain conditions, this is possible; it may thus be looked on as a uniqueness theorem. As to these conditions, we refer the reader to Milnor's notes on Differentiable Structures for spectacular counterexamples which occur when they are removed.

A weak diffeotopy often occurs in the following form: we are given a level preserving imbedding $h\colon M \times I \to N \times I$. We cannot immediately extend this to a normalised weak diffeotopy in the above sense, but we define $H\colon M \times \mathbb{R} \to N \times \mathbb{R}$ by

$$H(m,t) = (H_t(m), T) \qquad \text{where } H_t = h_{B_P(t)}.$$

H is clearly level-preserving, normalised, and an imbedding.

Proposition 4.2. *Weak diffeotopy is an equivalence relation.*

Proof. The definition $h(m,t) = (h_0(t), t)$ gives a weak diffeotopy between h_0 and itself. If h' gives one between h_0 and h_1, then h'', where $h''(m,t) = h'(m, 1-t)$ gives a weak diffeotopy between h_1 and h_0. Finally, let h', h'' be normalised weak diffeotopies between h_0 and h_1 and h_1 and h_2. Then set

$$h_t''' = \begin{cases} h'_{3t}(m) & \text{if } t \leqq 1/2 \\ h''_{3t-2} & \text{if } t \geqq 1/2; \end{cases}$$

this is a smooth imbedding, since h' and h'' are so, and we have $h_t''' = h_1$ for $1/3 \leqq t \leqq 2/3$, so that the two parts of the definition fit smoothly. □

One of our main objectives will be to determine the set of equivalence classes; in some simple cases this is accomplished in Part III.

Definition 4.3. The *support*of a diffeomorphism h of a smooth manifold N is the closure of the set of points P with $h(P) \neq P$.

The support of a weak diffeotopy h of M in N is the closure of the set of points $P \in M$ such that $h_t(P)$ is not independent of t.

Theorem 4.4. *Let M, N be smooth manifolds, perhaps with boundary, and let $h\colon M \times \mathbb{R} \to N \times \mathbb{R}$ be a weak diffeotopy of M in N, Suppose that the support K of h is compact, and contained in $\mathring{N}$. Then there is a diffeotopy k of N, whose support is compact and contained in $\mathring{N}$, which covers h; in particular , h is strong.*

We shall refer to this as the *Diffeotopy Extension Theorem.*

Proof. Since K is contained in $\mathring{N}$, we can ignore the boundary of N, and suppose simply that N is a smooth manifold, for if the result is proved in this case, the diffeotopy k of N which we obtain, having compact support, equals the identity on a neighbourhood of $\partial N \times \mathbb{R}$, and can therefore be extended to the boundary as the identity.

We shall prove the result by applying Part 0, Theorem 4.7 on 1-parameter groups of diffeomorphisms. In fact, let k be a diffeotopy of $N \times \mathbb{R}$, with compact support. Then k defines a vector field on $N \times \mathbb{R}$, for if X_0 is the vector field which projects to O on N and to $\partial/\partial t$ on $\mathbb{R}$, we define an associated vector field

X_k to k as $dk(X_0)$; since k is a diffeomorphism, this a one-valued vector field on $N \times \mathbb{R}$. Since k is level-preserving, its projection on the second factor is still $\partial/\partial t$. Also, as k has compact support, $X_k = X_0$ except at some points of a compact set.

Conversely, suppose given a vector field $X(= X_k)$ with these properties, that its projection on $\mathbb{R}$ is $\partial/\partial t$, and that it agrees with X_0 outside a compact set; we assert that k can be recovered. In fact, referring to Part 0, Theorem 4.7, note that X_0 is complete (as remarked after that theorem), hence also X, by Corollary Part 0. 4.9. Thus there is a 1-parameter group (φ_t) of diffeomorphisms of $N \times \mathbb{R}$. We set $\varphi_t(n, 0) = (k'_t(n), t) = k'(n, t)$; that the second components is t follows from our assumption on X. We now say that $k = k'$; this in fact follows from the local uniqueness in Part 0, Theorem 4.5, for k, k' each satisfy

$$\frac{\partial}{\partial t} x_i(k(m,t)) = X_i(k(m,t))$$

where the x_i are local coordinates in N, and the X_i the components of X in these coordinates.

We conclude that to construct the diffeotopy, it is sufficient to construct the vector field X. By the proof that $k = k'$, we see that the necessary and sufficient condition that k covers h is that on $h(M \times \mathbb{R})$, $X = dh(\partial/\partial t)$. Thus the problem is reduced to the construction of a vector field X on $N \times \mathbb{R}$ satisfying

i) $X = X_0$ outside a compact set.

ii) The projection of X on $\mathbb{R}$ is everywhere $\partial/\partial t$.

iii) On $h(M \times \mathbb{R})$, $X = dh(\partial/\partial t)$.

It is possible to carry out this construction more or less exactly, using tubular neighbourhoods, but to include the case of boundaries, we use rather more general method, already used above in proving existence of Riemannian structure. First, for convenience, let us give N a Riemannian metric and $N \times \mathbb{R}$ the product metric. Now condition ii) determines the component of X in the direction of $\mathbb{R}$ (in a fashion compatible with i), iii)); we must find the component in the direction of N. We assert that if we can do this in a neighbourhood of each point of $h(M \times \mathbb{R})$, X can be constructed. For such neighbourhoods, together with the complement of $h(M \times \mathbb{R})$, form an open covering of $N \times \mathbb{R}$. By Part 0, Theorem 2.1, we can find C.N.s $\varphi_\alpha\colon U_\alpha \to U(O, 3)$ refining this covering, and by the proof of its Corollary 2.2, a corresponding partition of unity $\{\Psi_\alpha\}$. If, then, a function X_α can be constructed in each set U_α to satisfy conditions i) - iii); we can define simply $X = \sum_\alpha X_\alpha \Psi_\alpha$, which will satisfy all the conditions.

Now $h(M \times \mathbb{R})$ is a submanifold of $N \times \mathbb{R}$, hence in a neighbourhood of any point of it we can find a C.N. $\varphi\colon U \to \mathbb{R}^{n+1}$ with $U \cap \mathrm{Im} h = \varphi^{-1}(\mathbb{R}^{m+1})$; say for simplicity that the image of U is $U(0, 1)$. Then $d\varphi(dh(\partial/\partial t)) = \sum a_i \partial/\partial x_i$ in $U(0, 1)$ in $\mathbb{R}^{m+1}$; we define X by taking the same formula in $\mathbb{R}^{n+1}$ (i.e., by taking the a_i independent of the last $n - m$ coordinates). In the case of boundaries, the a_i are only defined on the set in $\mathbb{R}^{m+1}_+$. But by Whitney's Extension Theorem, Part 0.4.4, they can be extended to smooth functions on $U(0,1)$ in $\mathbb{R}^{m+1}$, and then extended to $\mathbb{R}^{n+1}$ as above. This completes the proof of the result. □

Corollary 4.5. *If N is a smooth manifold, M a compact submanifold (perhaps with boundary), then any weak diffeotopy of the inclusion $i\colon M \subset N$ is strong.*

Corollary 4.6. *If N is a smooth manifold with boundary, any weak diffeotopy of a compact submanifold (perhaps with boundary) of $\mathring{N}$ is covered by a diffeotopy of N.*

Proof. By the theorem, it is covered by a diffeotopy of $\mathring{N}$ with compact support. Thus ∂N has a neighbourhood in $\mathring{N}$ let fixed by the diffeotopy, which can thus be extended to N, defining it to be fixed on ∂N. □

Proposition 4.7. *Any diffeotopy of ∂N is covered by a diffeotopy of N.*

Proof. We shall suppose the diffeotopy h_t of ∂N normalised so that $h_t = 1$ for $t \leqq 1/3$ and $h_t = h_1$ for $t \geqq 2/3$. Let $\psi\colon \partial N \times I \to N$ be a tubular neighbourhood of ∂N in N (such exist by Proposition 3.8). Then we define a covering diffeotopy k_t of N by

$$k_t = 1 \text{ outside } \mathrm{Im}\psi; \qquad k_t\psi(P,s) = \psi(k_{ts}(P), s)$$

where

$$k_{ts}(P) = \begin{cases} P & \text{for } s \geqq t, \\ h_{t-s}(P) & \text{for } t \geqq s. \end{cases}$$

Thus for $s = 0$, k_{ts} agrees with h_t, and for $s \geqq 2/3$, $k_{ts}(P) = P$, so that k is everywhere smooth, and does cover h. □

Theorem 4.8. *Let N be a manifold with boundary, M s submanifold (perhaps with boundary). Any weak diffeotopy of M in N with compact support is covered by a diffeotopy of N with compact support.*

Proof. First suppose M a submanifold. Let $h\colon M \times \mathbb{R} \to N \times \mathbb{R}$ be the weak diffeotopy. By Theorem 3.15, let $\psi\colon \partial N \times \mathbb{R} \times I \to N \times \mathbb{R}$ be a tubular neighbourhood of the boundary of $N \times \mathbb{R}$ whose restriction to $\mathrm{Im}h$ gives a tubular neighbourhood of the boundary of that. Now by Theorem 4.4, the weak diffeotopy of ∂M can be covered by one of ∂N. By Proposition 4.7, this is covered by a diffeotopy of N; moreover, by the construction of this diffeotopy, it covers the diffeotopy of M not only at ∂M, but in a neighbourhood, and has compact support.

This still fails to cover the diffeotopy of M, but only on a set of compact support, contained in $\mathring{N}$, and the methods of Theorem 4.4 now apply to complete the proof.

If M is a submanifold with boundary, there is a similar proof, using instead Corollary 6.3. □

We shall need one or two further kinds of diffeotopy extension, when we come to consider corners, but feel that by now proofs may be left to the reader. We mention one immediate application of our results.

Proposition 4.9. *Let N^n be a manifold (perhaps with boundary), M^m a compact submanifold with boundary. Then there is a submanifold U^m of N^n containing M^m.*

Proof. First suppose that N has no boundary. Let $\varphi\colon \partial M \times I \to M$ be a tubular neighbourhood of ∂M in M. We define a weak diffeotopy of M by

$$\begin{cases} h_t(P) = P & P \notin \mathrm{Im}\varphi \\ h_t\varphi(P,u) = \varphi(P, f(t,u)) \end{cases}$$

where f is chosen with $f(t,u) = u$ for $u > 1 - \varepsilon$, $f(0,u) = u$, $f(t,0) > 0$, for $0 < \varepsilon$ and $\partial f/\partial u > 0$ everywhere; so that the diffeotopy 'pushes' the boundary a little way into M. E.g., we can take

$$f(t,u) = u + B_P(t-u)$$

provided $t \leqq k$, where in this range $B'_P(t) < 1$. Now h_t is weak, so strong (M being compact), and covered by H_t, say, $h_k(M) \subset \mathring{M}$, so we can take $U = H_k^{-1}(\mathring{M})$.

If N is bounded, we argue similarly, using that part of the boundary of M not contained in ∂M. □

This result has the effect that to describe a neighbourhood of M in N, we can use tubular neighbourhoods of U; tubes round M do not give neighbourhoods.

Chapter 5

Tubular Neighbourhood Theorem

We shall now use our results on diffeotopy extension to complete the discussion in Chapters 2 and 3 of tubular neighbourhoods by showing that these are, essentially, unique. This enables us to pass form knowledge of the structure of a compact submanifold M^m of a manifold N^n to knowledge of a neighbourhood of M^m: the only extra piece of information needed is the structure of the normal bundle $\mathbb{N}(N/M)$. Thus our considerations help with the general problem of building up global results from merely local ones.

We recall the definition. If B is an $(n-m)$-disc bundle over M, with group $\mathbf{O}(n-m)$, and central cross-section B_0, then a tubular neighbourhood of M in N is an imbedding $\varphi\colon B \to N$, as submanifold with boundary, extending the projection of B_0 on M.

Definition 5.1. Two tubular neighbourhoods $\varphi\colon B \to N$ and $\varphi'\colon B' \to N$ are *equivalent* if there is a bundle map $\chi\colon B \to B'$ over the identity map of M, and a strong diffeotopy of φ on $\varphi'_0\chi$ which is fixed on B_0.

Our object is to show that any two tubular neighbourhoods are equivalent. Since we shall use the result of Chapter 4 we shall have to assume that M is compact. One would expect that this assumption was unnecessary; however, it cannot be simply omitted. A tubular neighbourhood of $\mathbb{R}^1$ in $\mathbb{R}^2_+$, not equivalent to standard ones:

Example 5.2. T is the set $|y| < 3$ and $x^2 + (y-2)^2 \geqq 1$ and the projection of T on $\mathbb{R}^1$ is defined by straight lines through $(0, 3)$. (See Figure 5.1).

Clearly this gives a tubular neighbourhood, equally clearly non-standard.

For applications in later parts, we shall usually assume all manifolds compact anyway.

Let $\varphi\colon B \to N$ be a tubular neighbourhood for M in N. We consider the bundle E associated to B but with fibre $\mathbb{R}^{n-m}$, and correspondingly extend the

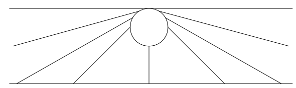


Figure 5.1: Non-standard tubular neighbourhood

group to $\mathbf{GL}(n-m)$. B is a submanifold with boundary of E. For the tubular neighbourhoods of Chapter 2 E is simply the normal bundle $\mathbb{N}(N/M)$.

Definition 5.3. An imbedding $\bar{\varphi}\colon E \to N$ as open submanifold, extending the projection of E_0 on M, is a ***weak tubular neighbourhood*** of M in N.

Lemma 5.4. *Any tubular neighbourhood $\varphi\colon B \to N$ can be extended to a weak tubular neighbourhood $\bar{\varphi}\colon E \to N$.*

Remember that we are assuming that M is compact.

Proof. We can define a weak diffeotopy of φ as follows. Recall that over each neighbourhood in M, B is a product of M with a vector space; in the sequel, we permit ourselves to form sums and products by scalars in these vector spaces, using the standard notation. Then our weak diffeotopy is $\varphi_t(m,v) = \varphi(m,tv)$ for $1/2 \leqq t \leqq 1$ (where $m \in M$, $v \in D^{n-m}$, the fibre). Since M, and so also B, is compact, the weak diffeotopy is strong: say it is covered by the diffeotopy k_t of N. But $\varphi^{1/2}$ can be extended to a weak tubular neighbourhood, e.g., by $\bar{\bar{\varphi}}$:

$$\bar{\bar{\varphi}}(m,v) = \varphi(m, \frac{\gamma(|v|)}{|v|} \cdot v)$$

where γ is smooth, $\gamma(t) = \frac{1}{2}t$ for $0 \leqq t \leqq 1$, $\gamma' > 0$, and $\gamma(t) < 1$. Such a γ may easily be constructed by using bump functions, e.g.,

$$\gamma(t) = \frac{1}{3}\int_0^t \{1 + (e^{-x} - 1)B_P(x-1)\}dx.$$

We can now define $\bar{\varphi} = k_{1/2}^{-1} \circ \bar{\bar{\varphi}}$. ☐

Lemma 5.5. *Let $\bar{\varphi}\colon E \to N$, $\bar{\varphi}'\colon E' \to N$ be weak tubular neighbourhoods of M in N such that* $\mathrm{Im}\bar{\varphi} \subset \mathrm{Im}\bar{\varphi}'$*. Then for some bundle map $\bar{\chi}\colon E \to E'$, there is a weak diffeotopy of $\bar{\varphi}$ on $\bar{\varphi}'\bar{\chi}$ which is fixed on B_0.*

Proof. Let $j = \bar{\varphi}'^{-1} \circ \bar{\varphi}\colon E \to E'$, then j is an imbedding. Consider the mappings $j_t\colon j_t(e) = t_j^{-1}(te)$ for $0 < t \leqq 1$, $e \in E$; where the multiplications by t^{-1}, t are again scalar multiplications in the fibre. Clearly $j_1 = j$; we shall show that the definition of j_t can be extended to $t = 0$, and that j_0 can be taken as $\bar{\chi}$:

$\bar\varphi' j_t$ will then give the required weak diffeotopy of $\bar\varphi = \bar\varphi' j$ on $\bar\varphi'\bar\chi$; it is clearly fixed on B_0.

Take local coordinates $x = (x_1, \ldots, x_m)$ in M, and let y, z be Euclidean coordinates in the fibres of E, E'. Then setting $j(x,y) = (\alpha(x,y), \beta(x,y))$ we have

$$j_t(x,y) = (\alpha(x,ty), t^{-1}\beta(x,ty)).$$

But j carries the zero cross-section of E onto that of E', so

$$\alpha(x,0) = x, \qquad \beta(x,0) = 0.$$

Now by Part 0, Lemma 3.3, applied to β (regarded as a function of y with x as a parameter), there are smooth functions β_i with

$$\beta(x,y) = \sum y_i \beta_i(x,y)$$

Then $t^{-1}\beta(x,ty) = \sum y_i \beta_i(x,ty)$, so we can write j_t in the form

$$j_t(x,y) = (\alpha(x,ty), \sum y_i \beta_i(x,ty))$$

where the left hand side is a smooth function also at $t = 0$. This shows that we have a smooth map $J\colon E \times I \to E' \times I$ defined by the j_t; to have a weak diffeotopy, we must check that the Jacobian is everywhere nonzero. This is clear for $t \neq 0$, since j is a diffeomorphic imbedding, and multiplication by t or t^{-1} gives a diffeomorphism. Now

$$j_0(x,y) = (x, \sum y_i \beta_i(x,0)) = (x, \sum y_i \frac{\partial \beta}{\partial y_i}|_{y=0})$$

induces a linear map of each fibre, with matrix $(\partial\beta_j/\partial y_i) = (\partial z_j/\partial y_i)$ which is also the matrix of partial derivatives of j on B_0. Since j_0 is an imbedding, this is nonzero. So j_0 is a fibre map, with each fibre mapped isomorphically, so is a homeomorphism; since the Jacobians are nonzero, it is a diffeomorphism (Lemma 2.5), and we can take $\bar\chi = j_0$. We have also verified by the same token that J is a weak diffeotopy. □

Corollary 5.6. *The result holds also without th assumption* $\mathrm{Im}\bar\varphi \subset \mathrm{Im}\bar\varphi'$.

Proof. For $\mathrm{Im}\bar\varphi \cap \mathrm{Im}\bar\varphi'$ is a neighbourhood of M, which thus has a tubular neighbourhood, hence also a weak one $\bar\varphi''$, with $\mathrm{Im}\bar\varphi'' \subset \mathrm{Im}\bar\varphi \cap \mathrm{Im}\bar\varphi'$. Then there are bundle maps modulo which $\bar\varphi''$ is weakly diffeotopic both to $\bar\varphi$ and $\bar\varphi'$, whence the result follows. □

Lemma 5.7. *Let* $\bar\varphi\colon E \to N$, $\bar\varphi'\colon E' \to N$ *be weak tubular neighbourhoods of* M *in* N *where the bundles* E, E' *have group* $\mathbf{O}(n-m)$. *Then the conclusion of Lemma 5.5 holds, with* $\bar\chi$ *an* $\mathbf{O}(n-m)$*-bundle map.*

Proof. It suffice to show that any $\psi\colon E \to E'$ which is a bundle map when the group is extended to $\mathbf{GL}_{n-m}(\mathbb{R})$ is weakly diffeotopic to an $\mathbf{O}(n-m)$-bundle map. As above, in coordinates, ψ is given by

$$\psi(x,y) = (x,z) \quad \text{where} \quad z_i = \sum a_{ij}(x)y_j.$$

Now since the group is the orthogonal group, we can speak of the length of a vector in the fibre (cf Part 0, Chapter 3). By the Gram-Schmidt orthogonalisation process, take the vectors b_i with components a_{ij}, and write $b_i = \sum_{j=1}^{i} \lambda_{ij} e_j$, where the e_i are orthonormal, and each $\lambda_{ij} > 0$. If e_i has components e_{ij}, consider now the weak diffeotopy

$$k_t(x,y) = (x, z_t), \quad \text{where} \quad (z_t)_i = \sum_{j,k} (t\lambda_{ij} + (1-t)\delta_{ij}) C_{jk} y_k)$$

That this is a weak diffeotopy follows as no matrix $(t\lambda_{ij} + (1-t)\delta_{ij})$ is singular (for the matrix is triangular, with nonzero diagonal terms); k_1 is the given map ψ, end k_0 takes one orthonormal base to another, so is an $\mathbf{O}(n-m)$-bundle map. □

Corollary 5.8. *Let $\varphi\colon B \to N$, $\varphi'\colon B' \to N$ be tubular neighbourhoods of M in N. Then there is a bundle map $\chi\colon B \to B'$, with $\varphi' \circ \chi$ weakly diffeomorphic to φ.*

Proof. By Lemma 5.4, φ, φ' extend to weak tubular neighbourhoods $\bar{\varphi}$, $\bar{\varphi}'$; by Lemma 5.4, there is a bundle map $\bar{\chi}\colon E \to E'$ with corresponding property. Then $\bar{\chi}$ maps B into B', and so we can take χ as its restriction. □

Corollary 5.9. *Under these conditions, B and B' are equivalent bundles.*

Proof. χ is a bundle isomorphism. □

Theorem 5.10. *(Tubular Neighbourhood Theorem) Let N^n be a smooth manifold and M^m a compact submanifold. Then any two tubular neighbourhoods of M in N are equivalent.*

Proof. This follows from Corollary 5.8 since, by Theorem 4.4, the weak diffeotopy we have constructed is in fact strong. □

As a first corollary, we obtain a useful little result.

Theorem 5.11. *(Disc Theorem) Let N be a connected manifold (perhaps with boundary), $f_1, f_2\colon D^n \to N^n$ imbeddings as submanifold with boundary. Then f_1 and f_2 are strongly diffeomorphic unless N is oriented and f_1, f_2 have opposite orientations.*

Proof. Let $P_i = f_i(O)$ $(i = 1, 2)$. Since $\mathring{N}$ is connected, by Definition 2.10 there is a smooth path connecting P_1 and P_2 in $\mathring{N}$, i.e., a weak diffeotopy of P_1 and P_2, considered as submanifolds of zero dimension. By the diffeotopy extension theorem, there is a strong diffeotopy. Hence we may suppose $P_1 = P_2 = P$. Now f_1, f_2 are tubular neighbourhoods of P, so by theorem 5.10, there is an orthogonal transformation χ of D^n, such that f_1 and $f_2 \circ \chi$ are strongly diffeotopic.

Now if $\chi \in \mathbf{SO}(n)$, then clearly f_2 is weakly, so also strongly diffeotopic to $f_2 \circ \chi$, so the result follows. If not, and N is orientable, we have the case excluded by the theorem. If N is non-orientable, there is an orientation reversing smooth path (of the discussion after Definition 3.9), and if we take P on a strong diffeotopy round such a path, the sign of the determinant of χ will change. □

We shall use numerous extension of Theorem 5.10 in the sequel; let us indicate one or two briefly here. The definition of equivalence remains the same.

Proposition 5.12. *Any two tubular neighbourhoods of ∂N in N are equivalent, if ∂N is compact.*

Proof. Follow the above closely. The analogues of Lemma 5.4 and Lemma 5.5 follow as before. In Lemma 5.7, note only that our group is not $\mathbf{GL}_1(\mathbb{R})$ or $\mathbf{O}(1)$, but simply $\mathbf{GL}_1^+(\mathbb{R})$ or $\mathbf{SO}(1)$ - the trivial group. This makes for a slight simplification in the argument. □

Proposition 5.13. *The result of Theorem 5.10 holds also if N has a boundary.*

We note that in proving uniqueness of tubular neighbourhoods, in contrast to the case where we had to prove existence in Chapter 3, no extra difficulties arise in the case where we have boundaries.

Chapter 6

Corners and Straightening

In this chapter we shall pay a little attention to manifolds with a corner, and give a process of straightening this, so as to have simply a manifold with boundary. This will be very useful later on, where any corners which occur may be ignored by the results of this chapter.

We first need existence and uniqueness theorems for a lot of new kinds of tubular neighbourhood. Let M be a manifold with corner $\angle M$. A *Riemannian structure* on M is defined as before, with the extra condition that the two parts of ∂M at a point of $\angle M$ meet orthogonally (i.e., the vectors normal to them are perpendicular). A *tubular neighbourhood* of ∂M is defined as before. However, $\partial M \times I$ does not have the structure of a smooth manifold (of any kind) on $\angle M \times I$, so we must interpret "imbedding" to mean a homeomorphism into, which is a diffeomorphism except on $\angle M \times I$, and with all partial derivatives continuous at $\angle M \times I$ from each side.

Lemma 6.1. *There exist s a tubular neighbourhood of ∂M in M, if ∂M is compact.*

Proof. First define inward-pointing vectors on ∂M; except on $\angle M$ these are, as usual, vectors $\sum \lambda_i \partial/\partial x_i$ with $\lambda_1 > 0$, in terns of a C.N. On $\angle M$, we require $\lambda_1 > 0$, $\lambda_2 > 0$. We observe that at each point, the space of inward-pointing vectors is convex. Now construct on ∂M a smooth field of inward-pointing vectors: we first do this everywhere locally, and piece together with a partition of unity (cf Part 0, proof of Theorem 3.8). The exponential map applied to this field now gives a local diffeomorphism, and from this we deduce a tubular neighbourhood as usual, using Part 0, Corollary 2.18 and Lemma 2.13. □

(We could do without compactness, but the result is not of sufficient importance to make it worth the trouble). Our next object is to obtain a tubular neighbourhood of $\angle M$ in M; this is of no little difficulty, and our first suggested proofs were fallacious. We hope the following is not. The tubular neighbourhood is as usual an imbedding of a fibre bundle. The choice of the fibre is of no importance, provided we do get a neighbourhood; we obtain a set of the form

$|x| \leqq y \leqq 1$ in $\mathbb{R}^2$, with group $\mathbb{Z}_2$ operating by reflection in the y-axis. This is somewhat more convenient than coordinates x_0, x_1.

Theorem 6.2. *If $\angle M$ is compact, there exists a tubular neighbourhood of $\angle M$ in M.*

Proof. We first suppose a Riemannian structure given on M, and take the vector field on $\angle M$ consisting of that normal vectors inclined at $\pi/4$ to each part of ∂M. As in Lemma 6.1, we can apply the exponential map to such vectors (provided they are inward-pointing), and for sufficiently small ones obtain a diffeomorphic imbedding of $\angle M \times I$.

Next we construct geodesics normal to this subset, until they meet the boundary ∂M. Observe that by the usual arguments, every point of a sufficiently small neighbourhood of $\angle M$ lies on just one geodesic. We use this to define a map of such a neighbourhood into $\mathbb{R}^2$. A point P in the image of $\angle M \times I$, at distance $\lambda\varepsilon$ from $\angle M$ (where ε is the "sufficiently small" distance,) is mapped to (O, λ). A point in a normal geodesic of P, at distance $\mu\varepsilon$ from $\angle M$ (where ε is the "sufficiently small" distance) is mapped to (O, λ). A point in a normal geodesic of P, at distance $\mu\varepsilon$ from it, is mapped to $(\pm\mu, \lambda)$. Here, the choice of sign is indeterminate, but can be made coherently locally.

By the usual arguments, our mappings are smooth (they come from the exponential map.) The product to $\angle M \times \mathbb{R}^2$ is thus also smooth, and has Jacobian 1 on $\angle M$, so is a local homeomorphism, and if ε is small enough, a diffeomorphism. Here I have been imprecise: as the map to $\mathbb{R}^2$ was only defined up to a reflection, my map really goes to an $\mathbb{R}^2$-bundle over $\angle M$, in general non-trivial.

The image in $\mathbb{R}^2$ is defined by equations of the type

$$-h(y)y \leqq x \leqq g(y)y, \qquad 0 \leqq y \leqq 1$$

where $h(0) = g(0) = 1$ (since the angle is right) and h, g are positive in the range under consideration, and depend also on the point of $\angle$. To simplify this, we define a new coordinate w by

$$2x = \{g(y) + h(y)\}w + \frac{1}{y}\{g(y) - h(y)\}w^2;$$

provided ε is small enough (for the last time!) this defines w as an increasing function of x, restricted only by $-y \leqq w \leqq y$.

Reflection in the y-axis interchanges g and h and changes the sign of x. Thus it also change the sign of w, and our bundle has a well-defined fibre and group. Finally, the new coordinate is also smooth; indeed this is quite clear from the definition above. □

We have left out most of the details in this proof to make the ideas clearer. The only other proof to my knowledge is in Cerf's thesis.

In the corollaries we shall suppose, for simplicity, that we can write $\partial M = \partial_1 M \cup \partial_2 M$, $\angle M = \partial_1 M \cap \partial_2 M = \partial\partial_1 M = \partial\partial_2 M$; so that $\angle M$ separates ∂M

into parts with closures $\partial_1 M$, $\partial_2 M$. This is the case for all the corners that we actually need. A tubular neighbourhood of $\partial_1 M$ is defined in the usual way; the image contains a neighbourhood of $\angle M$.

Corollary 6.3. *There exists a tubular neighbourhood of $\partial_i M$ in M.*

Proof. As in the proof of Proposition 3.11, we can use the tubular neighbourhood of $\angle M$ in M to construct a metric adapted to each of $\partial_1 M, \partial_2 M$ in a neighbourhood of $\angle M$. The construction of the tubular neighbourhood now proceeds as usual. □

Corollary 6.4. *There exists a metric adapted to ∂M.*

Proof. We use the tubular neighbourhoods of the above Corollary and the method of Proposition 3.11. Note that the product metrics given by these tubular neighbourhoods near the corner agree with the metric we have already (which was constructed using a tubular neighbourhood of $\angle M$); thus near $\angle M$ the metric is unaltered by this process. □

We observe that tubular neighbourhood theorems for the tubular neighbourhoods constructed in Theorem 6.2 and Corollary 6.3 follow without difficulty by the methods of Chapter 5; in contrast to the existence problem, we need no new idea here. We now turn to the main topic of the chapter. Let M be a manifold with compact corner.

Theorem 6.5. *There exists a manifold with boundary N such that there is a homeomorphism $h\colon M \to N$ which is a diffeomorphism except on $\angle M$. Moreover, there is a construction of such an N which gives a result unique up to diffeomorphism.*

Proof. Our construction is as follows. N will be M itself, with a different differential structure, defined by a new set of C.N.s. At points of $M \setminus \angle M$, the differential structure and C.N.s are unchanged. Let $\varphi\colon B \to M$ be a tubular neighbourhood for $\angle M$, where B is a bundle whose fibre is the set $|x| \leqq y \leqq 1$. Then a C.N. for $\angle M$, with coordinates $x_3, \ldots, x_m$ determines one for B, and so M, with additional coordinates x, y. We define N by the same mapping, followed by taking the new coordinate instead of y, $z = y^2 - x^2$. The C.N. is then defined locally by $z \geqq 0$, which is the right form for a manifold with boundary. $y = \sqrt{x^2 + z}$ is a smooth function of z except on $\angle M$, so the differential structure is unchanged elsewhere. Finally, as these C.N.s all come from a single tubular neighbourhood of $\angle M$, the differential structure so defined is clearly consistent.

The uniqueness up to diffeomorphism of such N follows at once from the tubular neighbourhood theorem for $\angle M$ in M. □

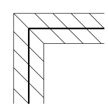
Definition 6.6. N is said to be derived from M by *straightening the corner.*

We reserve this term for this constructed N, not for any N which has an $h\colon M \to N$, a homeomorphism, diffeomorphic except on $\angle M$. Such N are in fact unique, but a proof of this would lie much deeper, since this allows arbitrary singularities of h on $\angle M$. We mention that the popular definition of straightening uses the same process, but replaces (x, y) by $(2xy, y^2 - x^2)$ instead of $(x, y^2 - x^2)$. The reason for our choice will soon be apparent.

Theorem 6.7. *Let $\varphi\colon \partial M \times I \to M$ be a nice tubular neighbourhood for ∂M in M. Let $\alpha\colon \partial M \to (0,1)$ be a map, smooth except on $\angle M$, and suppose $j\colon \partial M \to \mathring{M}$ defined by $j(P) = \varphi(P, \alpha(P))$ such that the image of j is a smooth submanifold ∂N. Such α exist, and if N is the interior of ∂N, i.e., the closure of that residual component of ∂N in M which does not contain ∂M, N is derived from M by straightening the corner.*

Remark 6.8. We need φ to be well-behaved near $\angle M$. It will suffice if φ is derived from a metric defined using a tubular neighbourhood of $\angle M$.

Proof. We shall first construct a homeomorphism h of N onto M, and then prove that it carries C.N.s for N onto those for M with the corner straightened.



Let us refer to the paths $\varphi(P \times I)$ as *orbits*. h will keep points outside Imφ fixed; those inside are moved along the orbits in such a way that a neighbourhood of $\varphi(P \times 1)$ is fixed, while $\varphi(P \times \alpha(P))$ is mapped to $\varphi(P \times 0)$. This may be effected as usual, using bump functions; the map can be made smooth away from $\angle M$.

Near $\angle M$ we take coordinates $(x, y, x_3, \ldots, x_m)$ as for a tubular neighbourhood. By assumption on φ, the orbits are obtained by letting y vary. Let $X = (x, 0, x_3, \ldots, x_m)$ be the corresponding point on the boundary. Then for X close to $\angle M$ and z small, we write

$$h(\varphi(X, \alpha(x) + z)) = \varphi(X, \sqrt{x^2 + z})$$

and use the bump function to pass smoothly from this to the other values of h. Observe that the coordinate x is well-determined up to sign referring to the tubular neighbourhood of $\angle M$. Finally, if $y = \sqrt{x^2 + z}$, $z = y^2 - x^2$ is indeed the coordinate introduced to straighten the corner. □

This theorem is very useful in reconciling the definition of straightening with the applications. For example, we have now

Corollary 6.9. *D^{r+s} is derived from $D^r \times D^s$ by straightening the corner.*

Proof. We can take the tubular neighbourhood of $\partial(D^r \times D^s)$, where $D^r \times D^s$ is imbedded in the standard way in $\mathbb{R}^{r+s}$, to be defined by orbits which are straight lines through O. Then the image of j can be taken as a sphere with centre at the origin. □

So far we have discussed straightening corners. We may also consider the converse process, the introduction of corners. For given a manifold with boundary N, and a submanifold L of ∂N of codimension 1, we can construct a tubular neighbourhood of L in N, and redefine the differentiable structure to introduce a corner along L. The resulting M is unique up to diffeomorphism, and if we straighten the corner, we return to N. The proof of these results are parallel to those above, but are much easier.

Proposition 6.10. *If L is a submanifold of ∂N of co-dimension 1, we can introduce a corner on L in an essentially unique way. If we straighten it, we recover L.*

Chapter 7

Cutting and Glueing

Cutting and glueing are simple geometrical constructions which, given some smooth manifolds (probably with boundaries and corners) and additional data where necessary, give rise to new manifolds. On account of their perspicuity, these methods were much used in the days of topology of surfaces, and they remain a very powerful tool.

We first discuss the simplest case of glueing. Let $M_i (i = 1, 2)$ be manifolds with boundary, $\partial M_i = Q_i$, and suppose given a diffeomorphism $h\colon Q_1 \to Q_2$ (the necessary additional data). We now form a smooth manifold. Take $M_1 \cup M_2$ (disjoint), and identify points corresponding under h. This gives a topological space N, and the identification map $\pi\colon M_1 \cup M_2 \to N$. Now take tubular neighbourhoods $\varphi_i\colon Q_i \times I \to M_i$. These define a map $\varphi_i\colon Q_1 \times D^1 \to N$ by

$$\varphi(q, t) = \begin{cases} \pi\varphi_1(q, t) & \text{if } t \geqq 0 \\ \pi\varphi_2(h(q), t) & \text{if } t \leqq 0; \end{cases}$$

these agree on $t = 0$ since Q_1 and Q_2 were identified using h. It is clear that φ is 1 - 1; in fact, a homeomorphism into. Now define a function f on N to be smooth provided $f \circ \pi$ is a smooth function on $M_1 \cup M_2$ and $f \circ \varphi$ a smooth function on $Q_1 \times D^1$. The axioms defining a smooth manifold are now clearly satisfied: C.N.s in M_1, $Q_1 \times D^1$, and in M_2 give rise to C.N.s in N, and where these overlap, they agree.

We have really not made full use of the assumption $\partial M_i = Q_i$, and none of the above argument is affected if ∂M_i is the disjoint union of a certain set of components, and Q_i the union of a subset of these components. In this case, the remaining boundary components form the boundary of N.

Definition 7.1. N is obtained by *glueing* M_1 to M_2 by h (or, along Q_1).

Proposition 7.2. *The manifold defined by glueing M_1 to M_2 by h is determined up to diffeomorphism, provided Q_1 is compact.*

Proof. The only arbitrary element in the definition was the choice of the tubular neighbourhoods Q_i. By the tubular neighbourhood theorem, these are unique up to diffeomorphism of M_i, so the result follows. □

It is unclear whether compactness of Q_1 is essential here. Certainly, glueing by inequivalent tubular neighbourhoods can give the same manifold as for example glueing two copies of $\mathbb{R}^2_+$, we always obtain a contractible 2-manifold, and any such is known to be diffeomorphic to $\mathbb{R}^2$ itself.

Definition 7.3. If N is obtained by glueing M to itself, via $1\colon \partial M \to \partial M$, we say it is defined by *doubling* M.

This particular case is useful in some contexts.

The inverse operation to glueing is cutting. Again, we discuss the simplest case first. Let N^n have Q^{n-1} as submanifold, and suppose that $N \setminus Q$ has just two components, with closures M_1 and M_2 so that $\partial M_1 = Q = \partial M_2$. It is immediate that each M_i is a submanifold with boundary of N, and has the induced structure of a smooth manifold. The M_i are uniquely determined by (N, Q) and N may have a boundary. No compactness is needed.

Proposition 7.4. *If N is defined by glueing M_1 to M_2 along Q_1, and we cut N along $\pi(Q_1)$, we recover M_1 and M_2. Conversely, if N^n and its submanifold Q^{n-1} are connected, Q separates N with parts M_1 and M_2 and we glue M_1 to M_2 along Q, then if Q is compact, we recover N.*

Proof. The first part is immediate from the definition of glueing. For the converse, if the above conditions are satisfied, we obtain M_1 and M_2. Now if $\varphi\colon Q \times D^1 \to N$ is a tubular neighbourhood of Q in N, φ defines by restriction tubular neighbourhoods of Q in M_1, M_2. If these are used in the glueing process, we clearly recover N. The second part of the result now follows from Proposition 7.2. □

Thus cutting and glueing are inverse operations. We now discuss cutting in a more general context. We continue to suppose that N^n is a smooth manifold (without boundary), Q a submanifold of unit codimension. However, we no longer suppose that Q separates N, or even that it separates a neighbourhood of Q; thus in general, when we cut N along Q, it will not fall into two pieces.

There are two quick ways of defining cutting. One is to let φ be a complete metric on N, and define M as the metric completion of $N \setminus Q$. A somewhat preferable procedure is to define M by deleting from N the interior of a tubular neighbourhood of Q; this has the advantage that M has a natural induced structure as submanifold with boundary. However it, like the firs proposal, makes use of additional structure - the tubular neighbourhood - which is not essential, and obscures the problem of uniqueness of the result; so we shall proceed differently.

Observe that, if $i\colon Q \to N$ is the inclusion, and $P \in Q$, then $di(Q_P)$ is a subspace of N_P of unit codimension, and so separates this real vector space into two components. We define a manifold M as follows. Its points are those of $N \setminus Q$, together with two for each point P of Q, one associated with each complementary component of $di(Q_P)$ in N_P or, as we shall say, *side* of Q in N. There is thus a natural projection $\pi\colon M \to N$. We take for C.N.s in M those induced by π from C.N.s in $N \setminus Q$; in addition, for each C.N. $f\colon U \to \mathbb{R}^n$ with

$f^{-1}(\mathbb{R}^{n-1}) = U \cap Q$ two C.N.s in M; induced by π from the restriction of f to the inverse images of $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$ (in the latter case, we must change the sign of the first coordinate to obtain a C.N. of standard type). Here, of course, the points of N corresponding to a certain side of Q in N are mapped by the C.N. for the corresponding side of $\mathbb{R}^{n-1}$ in $\mathbb{R}^n$; since df is nonsingular, it preserves the distinction between sides.

Definition 7.5. M is obtained by *cutting* N along Q.

We note that ∂M is a double covering of Q. In fact, it is easy to determine which covering.

Proposition 7.6. *Let Q^{n-1} be a submanifold of N^n, $\varphi\colon B \to N$ a tubular neighbourhood which extends to a weak tubular neighbourhood, M' the closure of $N \setminus \mathrm{Im}\varphi$, and M obtained by cutting N along Q. Then M is diffeomorphic to M', and hence ∂M to ∂B, the normal covering of Q in N.*

Proof. Cut B along Q (the zero cross-section). Then we obtain simply $\partial B \times I$: this is clear, since the whole is a bundle over Q with group $\mathbb{Z}_2$. Hence φ induces a tubular neighbourhood of the boundary of M, the complement of which is M'. It is now clear that M' is diffeomorphic to M; indeed, using the weak extension of φ, we can define a diffeotopy of the identity map of M' to a diffeomorphism onto M (cf proof of Lemma 5.4). The result follows. □

The corresponding extension of Proposition 7.4 for the present definition of cutting now follows. However, cutting is more general than simply the inverse of glueing as is clear, for example, when the normal covering of Q in N is non-trivial.

We shall need further generalisations of cutting and glueing which involve corners. If N is a manifold with boundary, Q a submanifold, we may define the manifold M obtained by cutting along Q precisely as above: the only new feature is that M has a corner at points corresponding ∂Q; this divides ∂M into two parts, corresponding respectively to ∂N and to Q.

Likewise, let $M_i (i = 1, 2)$ be manifolds with corners and let Q_i be part of the boundary of M_i with $\partial Q_i = \angle M_i$. Let $h\colon Q_1 \to Q_2$ be a diffeomorphism. Since the Q_i have tubular neighbourhoods by Lemma 2.3, we can define a manifold N by glueing M_1 to M_2 by h precisely as before; again the tubular neighbourhood theorem shows that if Q_1 is compact, the result is unique. The generalisation of propositions 7.4 and 7.6 to the present case now present no difficulty.

Finally we remark that it is sometimes desirable to glue together two parts of the boundary of the same manifold. If the parts are disjoint, we can use disjoint tubular neighbourhoods to effect this. If not, since it is usually the case that we are interested only in obtaining a result up to diffeomorphism, we can usually imitate the following trick. Let $\pi\colon \partial M \to Q$ be a double covering and suppose we wish to glue together points of ∂M lying above the same point of Q. Now the mapping cylinder B of π is a disc-bundle over Q, and so a smooth manifold with boundary, and the same result can be effected by glueing M to B

by the identity map of the boundary; that it is the same follows by Proposition 7.6.

As an important application of cutting, we mention the following.

Definition 7.7. Let M_1^m, M_2^m be connected smooth manifolds, $f_i\colon D^m \to M_i^m$ imbeddings. Delete the interior of the image of the f_i, and glue the result along the boundary $f_i(S^{m-1})$ by $f_2 f_1^{-1}$. The result is called the *connected sum*, written $M_1 \# M_2$. (It is obvious that it is connected).

Theorem 7.8. *$M_1 \# M_2$ is determined up to diffeomorphism by summands, unless these are both orientable, when there are two determinations.*

Proof. By the Disc Theorem 5.11, the imbeddings f_i are unique up to strong diffeotopy, and a possible change of orientation. By Proposition 7.2 the result of glueing, given f_1 and f_2, is unique up to diffeomorphism. Hence the result follows, except for considerations of orientation. Note that if f_1,f_2 are replaced by $f_1 \circ r$,$f_2 \circ r$, where r is a reflection, the connected sum is unaltered. Now if neither M_i is orientable, the result is trivial: if only M_2 is orientable, using the above possibility of simultaneous reversal, uniqueness again follows. If both are orientable, the result now has two possible cases. □

To make the result precise in the orientable case, we suppose the M_i both oriented, and that one of the f_i preserves, the other reverses orientation. The result is then again unique, and has a canonical orientation inducing the given ones of the M_i.

The connected sum is also defined for manifolds with boundaries and corners; we simply suppose that the f_i map into the interior. However, in this case we also have a different sum operation. Let us suppose that M_1^m, M_2^m are connected manifolds with connected boundaries. Let $f_i\colon D^{m-1} \to \partial M_i^m$ be an imbedding. Introduce a corner along $f_i(S^{m-2})$. We may now glue the $f_i(D^{m-1})$ together by $f_2 f_1^{-1}$

Definition 7.9. The result is called the *sum* $M_1 + M_2$ of M_1 and M_2.

Proposition 7.10. *$M_1 + M_2$ is determined up to diffeomorphism by M_1 and M_2 unless ∂M_1 and ∂M_2 are both orientable, when there are two sums.*

Proof. This follows by the Disc Theorem exactly as for Theorem 7.7. □

We conclude by summing up the simple properties of those operations.

Proposition 7.11. $M^m \# S^m \cong M^m, M^m + D^m \cong M^m, \partial(M_1 + M_2) = \partial M_1 \# \partial M_2.$

Proof. To form $M^m \# S^m$ we simply delete one disc from M^m, and replace it by another, equally good one.

The second result may be seen as follows. D^m is obtained from $D^{m-1} \times I$ by straightening the corner. Derive N from M by introducing a corner along $f(S^{m-2})$ as above; then glueing on $D^{m-1} \times I$ does not affect N other than by a diffeomorphism (as $f(D^{m-1})$ has a tubular neighbourhood by Corollary 6.3 and we have the usual deformation argument). The result follows by straightening the corners.

The last part is merely an observation of what happens to the boundary, for the sum operation; the proof is immediate. □

Part II

Theorems of Transversality and General Position

Chapter 0

Nul Sets

We now need a few standard facts about nul sets (i.e., sets of Lebesgue measure zero) which will be very useful in the sequel.

Definition 0.1. A subset A of $\mathbb{R}^n$ is *nul* if for each $\varepsilon > 0$, it can be enclosed in a countable union of balls of total volume $< \varepsilon$.

It is trivial that a countable union of nul sets is nul. Also that a nul set has no interior: its complement is everywhere dense.

Lemma 0.2. *Suppose U open in $\mathbb{R}^n$, $f\colon U \to R^n$ smooth, and $A \subset U$ nul. Then $f(A)$ is nul.*

Proof. Let K be a compact subset of U. Then in K the partial derivatives of f of first order are bounded, so infinitesimal lengths are multiplied by a bounded factor: let N be a bound. Then the image of a ball of radius r is contained in a ball of radius Nr; thus if B is contained in a number of balls in K of total volume less than ε, $f(B)$ is contained in a union of balls of total volume less than $N^n\varepsilon$.

Now as in Part 0, 2.1, we may find a countable set of discs $\bar{U}(x_i, 2\delta_i)$ contained in U, with the $U(x_i, \delta_i)$ covering U. Then if $A_i = A \cap U(x_i, \delta_i)$, we can cover A_i by balls contained in $\bar{U}(x_i, 2\delta_i)$ of total volume less than ε_i; hence by the above, $f(A_i)$ by balls of total volume less than $N_i^n\varepsilon_i$. Thus $f(A_i)$ is nul, and so is the countable union $f(A)$. □

Corollary 0.3. *Suppose U open in $\mathbb{R}^n$, $m < n$, $f\colon U \to \mathbb{R}^n$ smooth. Then $f(U)$ is nul.*

Proof. Define $F\colon U \times \mathbb{R}^{n-m} \to \mathbb{R}^n$ by $F(x, y) = f(x)$. Then $f(U) = F(U \times O)$, but clearly $U \times O$ is nul in $\mathbb{R}^n$. □

Definition 0.4. Let N^n be a smooth manifold. $A \subset U$ is *nul* if for each C.N. $\varphi\colon U \to \mathbb{R}^n$, $\varphi(U \cap A)$ is nul.

Since by the lemma, nul sets are preserved by smooth maps, it is sufficient to verify the condition for a set $(U_\alpha, \varphi_\alpha)$ of C.N.s with the U_α covering N.

Proposition 0.5. *Suppose $A \subset N_1^n$ be nul, and $f\colon N_1^n \to N_2^n$ be smooth. Then $f(A)$ is nul.*

Proof. The result follows at once from Lemma 0.2 and the definition. □

Corollary 0.6. *Suppose $m < n$, $f\colon M^M \to N^n$ be smooth. Then $f(M)$ is nul.*

Proof. As for Corollary 0.3. □

These give the basic properties of nul sets: we now go on to the deeper result which we shall need.

Definition 0.7. Let $f\colon N^m \to V^v$ be smooth. A point $P \in M$ is a *regular point* of f if $df\colon M_P \to V_{f(P)}$ has rank v. Otherwise P is a *critical point*, and $f(P)$ a *critical value* of f.

Theorem 0.8 (Sard's theorem)**.** *Let $f\colon M^m \to V^v$ be a smooth map. Then the set of critical values of f is nul.*

Proof. We observe that it is sufficient to consider values in a C.N. of V, and further that, since M is a countable union of C.N.s, we may also restrict attention to a C.N. of M. This reduces the proof to the case $V = \mathbb{R}^v$, M an open subset of $\mathbb{R}^m$. Now for $m < v$, the result follows by Corollary 0.3.

We give the proof here only for $m = v$. For $m > v$, we refer the reader to the paper by A. Sard, Bulletin of the American Mathematical Society 48 (1942) pp. 883-890.

Let P be a critical point. Since $m = v$, the Jacobian determinant of f vanishes at P, so given δ, we can find a ball containing P with $J(f) < \delta$ in the ball. Hence the volume of the image is $\leqq \delta\times$ volume of original ball: it can be contained in balls of at most twice this total volume.

If K is a compact submanifold of $\mathbb{R}^m$, A the set of critical points in K, we enclose these in small balls of total volume less than $2\mu(K)$, say. Then $f(A)$ can be enclosed in balls of total volume less than $4\delta\mu(K)$. But δ is arbitrarily small, so $f(A)$ is nul. The set of critical values is a countable union of sets $f(A)$, hence also nul. □

Chapter 1

Whitney's Imbedding Theorem

We open our discussion of the deeper properties of smooth manifolds with Whitney's imbedding theorem for two reasons. The first is historical: smooth manifolds were originally considered as submanifolds of Euclidean spaces, and this theorem reconciled this approach with the abstract form of definition which we prefer. Secondly, the proof is quite simple, and opens the way to our later discussion of the general transversality theorem.

Theorem 1.1. *Any compact manifold M^m (perhaps with boundary) can be imbedded in a Euclidean space.*

Proof. If the manifold is bounded, double up: any imbedding of the double restricts to give an imbedding of the original manifold. Now let $\varphi_i \colon U_i \to U(O,3)$ be the C.N.s constructed in 2.1, Part 0: since they are locally finite, and M compact, there are only a finite number. Also as in 2.2, Part 0, let $\Phi_i(P) = B_P(2 - |\varphi_i(P)|)$ for P in the range of φ_i, 0 otherwise. Now define functions f_{ij} by

$$\begin{aligned} f_{i0}(P) &= \Phi_i(P) \\ f_{ij}(P) &= \Phi_i(P) x_j(\varphi_i(P)) \quad P \quad \text{in range of} \quad \varphi_i \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Clearly, the f_{ij} are all smooth functions of P; if the range of i is $1 \leqq i \leqq N$, there are $(m+1)N$ of them, so they define a smooth map

$$F \colon M^m \to \mathbb{R}^{(m+1)N}$$

We assert that F is an imbedding: by 2.6, Part I, it is sufficient to prove that F is $1-1$ and an immersion (M being compact).

First, since the $\varphi_i^{-1}(U(O,1))$ cover M, each $P \in M$ belongs to at least one such. But in this set, $\Phi_i = 1$, $f_{ij}(P) = x_j(\varphi_i(P))$, and so these df_{ij} form a basis for M_P^*. Thus $df \colon M_P \to \mathbb{R}^{(m+1)N}_{f(P)}$ is $1-1$, and so F is an immersion.

Now if $F(P) = F(Q)$, and $P \in \varphi_i^{-1}(U(O,1))$, then $1 = \Phi_i(P) = f_{i0}(P)$, and so $1 = f_{i0}(Q) = \Phi_i(Q)$, and $Q \in \varphi_i^{-1}(U(O,1))$ also. But in this set, we can take the $f_{ij}(= x_j)$ as coordinates - since these have the same values for P and Q, then $P = Q$. Thus F is also $1-1$. □

This is the first of Whitney's theorems: the proof is very simple, but the result is rather weak. We shall now obtain a stronger version, with a bound on the dimension of the Euclidean space, and an approximation clause. It is also possible by similar methods to give a proof for non-compact manifolds; for us, it will be more convenient to defer this extension till we have the transversality theorem.

Each vector in $\mathbb{R}^n$ determines the parallel unit vector from the origin, and hence its end-point, which lies on S^{n-1}.

Lemma 1.2. *Let $f\colon M^m \to \mathbb{R}^n$ be an imbedding. Then the set of points of S^{n-1} whose vectors are parallel to a tangent of M^m is nul, if $n \geqq 2m+1$, and the set whose vectors are parallel to a chord is nul, if $n \geqq 2m+2$.*

Proof. Any tangent of M^m is parallel to a unit tangent. Let B be the subbundle of $\Pi(M)$ consisting of unit vectors. Then $df\colon \Pi(M) \to \Pi(\mathbb{R}^n)$ defines $df\colon B \to \Pi(\mathbb{R}^n)$, and since all tangent spaces to $\mathbb{R}^n$ have been identified with $\mathbb{R}^n$, there is a smooth map $\Pi\colon \Pi(\mathbb{R}^n) \to \mathbb{R}^n$. Moreover, since B consists of unit vectors, $\Pi \circ df$ maps B in S^{n-1}. Hence the set of points in S^{n-1} whose vectors are parallel to a tangent of M is the image of B under a smooth map. Since B has dimension $2m-1$, the first result follows from Corollary 0.6.

For chords we proceed similarly. Let $M \times M$ be the product manifold, $\Delta(M)$ the diagonal, and consider $C = M \times M \setminus \Delta(M)$: this is a smooth manifold. Since f is an imbedding, any two distinct points have distinct images, so if we define $f_1\colon C \to \mathbb{R}^n$ by $f_1(P,Q) = f(P) - f(Q)$ (vector subtraction), the image does not contain O. Thus we can normalise the image and define $f_2\colon C \to S^{n-1}$. Again we see that the set of points of S^{n-1} whose vectors parallel to a chord of M is the image under a smooth map; this time of C. Since C has dimension $2m$, the result follows as before. □

Theorem 1.3 (Whitney's Imbedding Theorem)**.** *Let M^m be a smooth compact manifold. Any map of M^m to $\mathbb{R}^{2m+1}$ may be approximated arbitrarily by an imbedding.*

Since we have not yet discussed topologies for mapping spaces (see Chapter 3 below), approximation is here to be understood in the sense of point-wise convergence.

Proof. Let $f_1\colon M^m \to \mathbb{R}^{2m+1}$ be the given map, $f_2\colon M^m \to \mathbb{R}^n$ some imbedding (which exists by Theorem 1.1). Consider the product map $f_3\colon M^m \to \mathbb{R}^{2m+1+n}$: this is an imbedding. For since f_2 is an immersion and $1-1$, so is f_3. Now by Lemma 1.2, the set $\mathcal{E}$ of points of S^{2m+n} whose vector is parallel to a tangent or chord is nul, thus its complement is everywhere dense. We choose a point x, close to the unit point on the last axis, and not in $\mathcal{E}$. Now project $f_3(M)$ in the direction x to $\mathbb{R}^{2m+n}$. Clearly the firs $2m+1$ coordinates of the projected map

f_4 differ from those of f_3, and hence of f_1, by an amount which can be made arbitrarily small by choice of x.

We say that f_4 is an imbedding. For since x is parallel to no chord of $f_3(M^m)$, no two distinct points of M have the same image under f_4; and since x is parallel to no tangent vector, there is no tangent vector which is mapped to zero by df_4. Thus f_4 is an immersion and $1-1$, hence an imbedding.

We may now repeat the projection process a further $(n-1)$ times, obtaining ultimately an imbedding in $\mathbb{R}^{2m+1}$ with coordinates differing by arbitrarily little from those of f_1. □

Theorem 1.4. *Any map of a compact M^m to $\mathbb{R}^{2m}$ may be approximated by an immersion.*

Proof. As for Theorem 1.3, we obtain an imbedding in $\mathbb{R}^{2m+1}$, and then choose $x \in S^{2m}$, arbitrarily close to the unit point on the last axis, and parallel to no tangent vector (which is possible, as before, using Lemma 1.2) Projecting parallel to x, we obtain the desired immersion. □

Chapter 2

Existence of Non-degenerate Functions

At a later stage in these seminars we shall give a method for describing compact manifolds up to diffeomorphism. The method consists in defining a smooth function $f\colon M^m \to \mathbb{R}$; and then we can regard M as "filtered" by the subset $f^{-1}(-\infty, a]$ as a increases. In order to carry out this process in detail, it is necessary to suppose f is non-degenerate.

Let f be a smooth function on M, and P a critical point of f, so that $df(M_P) = 0$. If we take local coordinates with P as origin, we have $f(O) = 0$ and $\partial f/\partial x_i$ vanishes at O for $1 \leqq i \leqq m$. It is now natural to consider the Hessian matrix $\partial^2 f/\partial x_i \partial x_j$ of second derivatives of f at O. We regard the Hessian as a symmetric bilinear form $H(f)\colon M_P \times M_P \to \mathbb{R}$, where

$$H(f)(\sum a_i \frac{\partial}{\partial x_i}, \sum b_i \frac{\partial}{\partial x_i}) = \sum a_i b_i \frac{\partial^2 f}{\partial x_i \partial x_j}$$

in local coordinates. Abstractly, if $u, v \in M_P$, we extend v to a local vector field $\underline{v}$ defined (at least) in a neighbourhood of P; then

$$H(f)(u, v) = u(\underline{v}(f)).$$

(Recall that a tangent vector is a mapping of functions on M to the reals, and hence a vector field maps functions to functions). This is independent of the extension $\underline{v}$ of v (since P is a critical point), and is clearly the same as the definition by coordinates.

Definition 2.1. P is a *degenerate (resp. non-degenerate) critical point* of f if $H(f)$ is a singular (resp. non-singular) bilinear form, f is-non-degenerate if it has no degenerate critical point.

Now suppose given an imbedding $i\colon N \to \mathbb{R}^n$. then since we identify $\Pi(\mathbb{R}^n)$ with $\mathbb{R}^n \times \mathbb{R}^n$, we may identify $\mathbb{N}(\mathbb{R}^n/M)$ with the submanifold of $\mathbb{R}^n \times \mathbb{R}^n$ given by pairs $\{(P, v) : P \in M, v$ orthogonal to $d_i(M_P)\}$. Recall that the exponential map is given by $\exp(P, v) = P + v$ (vector addition).

Definition 2.2. Let M be a submanifold of the complete Riemannian manifold N. Then a critical value of $\exp : \mathbb{N}(N/M)$ is called a *focus* of M; if the corresponding critical point is a vector at P, it is a focus of M at P.

We observe that by Sard's theorem, the set of foci of M in N (or in $\mathbb{R}^n$) is nul. It is then clear that the existence of non-degenerate functions will follow from the theorem below. For $P \in \mathbb{R}^n \setminus M$, define $L_P : M \to \mathbb{R}^1$ by $L_P(Q) = |P - Q|$.

Theorem 2.3. *L_P has a critical point at $Q \in M$ if and only if $\overrightarrow{PQ}$ is normal to M at Q. Q is a degenerate critical point if and only if P is a focus of M at Q.*

Proof. The first statement is clear. For the second, first suppose M is a curve in $\mathbb{R}^2$. Then a focus must be a point of intersection of consecutive normals, i.e., a centre of curvature. But L_P has a degenerate critical point at Q if and only if $|P - X|$ is constant to the second order at $X = Q$, i.e., again if and only if P is the centre of curvature of M at Q.

For general M, the argument is a little more complicated. Suppose that $P = Q+v$ is a focus, i.e., a singular point of exp at (Q, v). Then for a consecutive point $(Q + \delta Q, v + \delta v)$ in some direction, the difference $\delta Q + \delta v$ is of the second order of small quantities. Now since P is on a normal at Q, L_P has a critical point at Q, so $dL_P : M_Q \to \mathbb{R}^1$ is zero. But at $Q + \delta Q$, to the first order P again lies on the normal, and $dL_P : M_{Q+\delta Q} \to \mathbb{R}^1$ is zero. Thus if u is the tangent vector at Q corresponding to δQ, $u(\underline{v}(L_P)) = 0$ at Q for any $v \in M_Q$ i.e., $H(L_P)(u, v) = 0$ for all v, and $H(L_P)$ is singular on M_Q, so Q is a degenerate critical point of L_P.

If we suppose conversely that Q is degenerate, we can reverse the argument. Since $H(L_P)$ is singular, there exists u with $H(L_P)(u, v) = 0$ for all $v \in M_Q$, so $dL_P : M_{Q+\delta Q} \to \mathbb{R}^1$ vanishes to the first order if we move in the direction u, so to that order, P also lies on a normal at $Q + \delta Q$, and hence P is a focus of M at Q. □

Corollary 2.4. *Any compact manifold M admits non-degenerate functions.*

Proof. By Theorem 1.1, M can be imbedded in Euclidean space, by Sard's theorem, the set of foci (critical values of a smooth map) is nul, so we can choose $P \notin M$ not a focus, and then by the Theorem, L_P is a non-degenerate function. □

We remark that compactness is inessential, and also that using the approximation clause in Theorem 1.3, we could obtain one here. Also the condition $P \notin M$ is irrelevant; however, we should replace $L_P = |P - Q|$ by $|P - Q|^2$ in this case; P itself will then be a non-degenerate critical point. We shall obtain very precise forms of this corollary later, even specifying the needed number of critical points.

Chapter 3

Jet Spaces and Function Spaces

We now approach th general transversality theorem; for this we need a number of preliminary notions. We first discuss jets.

Lemma 3.1. *Let* $f\colon \mathbb{R}^v \to \mathbb{R}^m$ *be a smooth map such that* f *and all its partial derivatives of orders* $\leqq r$ *vanish at* O. *Let* φ, ψ *be diffeomorphisms of* $\mathbb{R}^v$, $\mathbb{R}^m$ *keeping* O *fixed. Then* $\psi f \varphi$ *has all partial derivatives of orders* $\leqq r$ *zero at* O.

Proof. The result is an immediate consequence of the chain rules for differentiating "a function of function". □

Clearly, also, the result holds if the maps are only locally defined, and writing $f = g - h$, holds also if we speak of g, h having equal derivatives rather than of f having zero ones.

Definition 3.2. Let $g, h\colon V^v \to M^m$ be smooth maps, and let $P \in V$. Define $g \sim_r h$ at P if, with respect to some local coordinate at P and $g(P)$, we have $g(P) = h(P)$, and all partial derivatives of order $\leqq r$ of g and h at P agree.

By the lemma, this is independent of the chosen coordinate system. Clearly, $\sim_r$ is an equivalence relation for maps defined on a neighbourhood of P. An equivalence class is called an *r-jet* of maps from V to M at P. The set of all jets of maps of V to M is the *jet space* $J^r(V, M)$.

Each jet is a jet of a map at some $P \in V$, so there is a natural projection $\pi_1\colon J^r(V, M) \to V$. Similarly (since $r \geqq 0$), since two functions g, h with the same r-jet at P, have $g(P) = h(P)$, there is another projection $\pi_2\colon J^r(V, M) \to M$. In fact it is clear that for $r = 0$ (when derivatives do not come in to it) we have $J^0(V, M) \cong V \times M$; here we may define a topology and the structure of a smooth manifold on the jet space using that on the product.

More generally, consider r-jets of functions f on a neighbourhood of P with $f(P) = Q$. With respect to local coordinates at P, Q, since two functions with the same partial derivatives define the same jet, we may take such partial derivatives as coordinates in $J^r(V, M)$. We need a streamlined notation. Let $(x_1, \cdots, x_v)$ be a set of local coordinates at P, $(y_1, \cdots, y_m)$ be a set of local coordinates at Q. We write $\omega = (\omega_1, \cdots, \omega_v)$ for an arbitrary set of non-negative integers; x^ω for $(x_1^{\omega_1} \cdots x_v^{\omega_v})$, $\partial_\omega = (\partial/\partial x_1)^{\omega_1} \cdots (\partial/\partial x_v)^{\omega_v}$, $|\omega| = \omega_1 + \cdots + \omega_v$, and $\omega! = \omega_1! \cdots \omega_v!$. Then if f is a function on a neighbourhood of P, $f(P) = Q$, its partial derivatives of order $\leqq r$ are simply the numbers $u_{\omega,j} = \partial_\omega y_j$ $(0 \leqq |\omega| \leqq r, 1 \leqq j \leqq m)$, thus these values determine the r-jet of f at P.

Conversely, given a set of numbers $a_{\omega,j}$ (where the point $(a_{0,j})$ must lie in the prescribed neighbourhood of Q), there exists a corresponding function - in fact, the polynomial

$$y_j = \sum a_{\omega,j} x^\omega / \omega!$$

Hence the set of r-jets $\mathfrak{J}$ with $\pi_1(\mathfrak{J}) = P$, $\pi_2(\mathfrak{J}) = Q$ is isomorphic to a Euclidean space.

If we now take $(x_i, U_{\omega,j})$ as local coordinate system in $J^r(V, M)$ - which we have seen o be possible - it is easy to convince oneself that coordinate changes are smooth (they exhibit, again, the chain rule for partial differentials): we shall spare the reader a detailed exhibition of them. We conclude that $J^r(V, M)$ is a smooth manifold.

We now observe that the projections π_1 and π_2 are smooth maps. Also, let $f\colon V \to M$ be a smooth map. Then at each $P \in V$ the equivalence class of f is our r-jet at P, so f defines a cross-section $\bar{f}\colon V \to J^r(V, M)$, which is smooth since f (and hence all its partial derivatives) is. Here it is useful to really restrict ourselves to infinitely differentiable maps - the condition was not essential in the preceding chapters. In the case $r = 0$, of course, $\bar{f}$ is just the graph of f; we may consider our case as generalised from this.

We now use the jet space terminology to discuss spaces of maps. Write M^V for the set of smooth maps of V in M: we wish to give this set a topology. First suppose V compact. Now each jet space $J^r(V, M)$ is a smooth manifold, so admits a complete Riemannian metric φ'^r: we shall replace by the non-Riemannian metric $\varphi^r = \inf(\varphi'^r, 1)$, which gives the same topology. Then if $f, g\colon V \to M$ are smooth maps, we define

$$\varphi^r(f, g) = \sup_{P \in V} \varphi^r(\bar{f}(P), \bar{g}(P)) \qquad \text{(this is finite since } V \text{ is compact)}$$

If we used φ^r to define a topology, we should obtain the topology of uniform convergence of f (with its first r derivatives). Instead we take $\varphi(f, g) = \sum_r 2^{-r} \varphi_r(f, g)$ to define a topology - here, convergence is equivalent to simultaneous convergence of f with all derivatives. Hence we may reasonably call it the smooth topology.

If V is not compact, (in fact in general), we define

Definition 3.3. The *smooth topology* on M^V is the topology of uniform convergence of all derivatives on compact subsets.

Lemma 3.4. *The smooth topology is metric.*

Proof. We know this is so if V is compact. If not, write $V = \cup_{i=1}^{\infty} V_i$ as a countable union of compact submanifolds (with boundary, but that is irrelevant) - say discs. Then the topology for M^{V_i} is defined by a metric φ_i, bounded by 1. Hence the metric $\varphi = \sum_{i=1}^{\infty} 2^{-i}\varphi_i$ defines the product topology on $\Pi_i M^{V_i}$, and hence the required topology on the subset M^V. □

Theorem 3.5. *With the smooth topology, M^V is a complete metric space.*

Proof. We have established that this topology is metrisable. Now again first suppose V compact. A Cauchy sequence in M^V must *à fortiori* be Cauchy with the metric φ^r. Since $J^r(V, M)$ is complete, the maps $\bar{f}_i$ converges to a limit $\bar{f}^r$, which is continuous, since the convergence was uniform.

Now for the $\bar{f}_i$, the coordinates $U_{\omega,j}$ are the partial derivatives of the $U_{0,j}$. Let ω' be derived from ω by increasing ω_i by unity, and $|\omega'| \leqq r$: then $U_{\omega',j} = \partial U_{\omega,j}/\partial x_i$ and so $U_{\omega,j}$ is the indefinite integral with respect to x_i of $U_{\omega',j}$. Integration commutes with uniform limits, so the same holds for $\bar{f}^r$. We deduce that for $\bar{f}^r$, $U_{\omega',j} = \partial U_{\omega,j}/\partial x_i$ again, so that the $U_{0,j} = y_j$ are r-times continuously differentiable. But this shows that $\bar{f}^r$ is the graph of an r-times differentiable function f, clearly independent of r, so f is smooth, and is the limit of the sequence.

If V is not compact, we write $V = \cup V_i$, and then M^V as a closed subset of the complete $\Pi_i M^{V^i}$ which is also complete. □

It follows that Baire's theorem applies to the space M^V (2.21 , Part 0).

Corollary 3.6. *The intersection of a countable family of dense open subsets of M^V is still dense.*

This is an exceedingly useful result.

Chapter 4

The Transversality Theorem

Let V^v, M^m be smooth manifolds, and let N^n be a submanifold of M^m. Let $f\colon V \to M$ be a smooth map.

Definition 4.1. The map f is *transverse* to N if for every $P \in V$ with $f(P) = Q \in N$, $df(V_P) + N_Q = M_Q$.

This may also be interpreted as stating that df induces an epimorphism of V_P on M_Q/N_Q, or equivalently, if $N_Q^{\perp}$ is the normal space to N at Q (the annihilator of N_Q in M_Q^* - see Definition Part I, 2.9), that df induces a monomorphism of $N_Q^{\perp}$ into V_P^*.

If $\dim V < \operatorname{codim} N$, the above condition cannot be satisfied: in that case transversality requires $f(V)$ to be disjoint from N.

The following result gives some indication of the geometrical meaning of the condition.

Lemma 4.2. *Let $f\colon V \to M$ be transverse to a submanifold N of M. then $f^{-1}(N) = W$ is a submanifold of V, whose codimension equals that of N in M.*

Proof. Let $P \in V$, $f(P) = Q \in N$, and let N be locally defined at Q by $x_1 = \cdots = x_c = 0$, where the x_i have linearly independent differentials at Q, and $c = \operatorname{codim} N$. Then by transversality, the functions $x_1 \circ f, \cdots, x_c \circ f$ have linearly independent differentials at P, and clearly their vanishing defines W near P. The result follows by the proof of 2.5, Part I (using 4.2, Part 0). □

We extend the concept as follows. Let N be a submanifold of $J^r(V, M)$. Then we say that f is transverse to N if $\bar{f}$ is so. Then roughly speaking, the transversality theorem states that almost any map is transverse to N. This is very general, so we need a lot of apparatus: we develop all the local results in a lemma.

Lemma 4.3. *Let $f\colon V^v \to M^m$ be a smooth map with graph $\bar{f}\colon V^v \to J^r(V, M)$, and let N be a submanifold of $J^r(V, M^)$ of codimension p. Let $\bar{f}(P) = Q \in N$. Then we can find* *i) a C.N.* U_1 *of* P *in* V*,*

ii) a C.N. U_2 *of* Q *in* $J^r(V, M)$ *and*

iii) an open neighbourhood W *of* f *in* M^V

such that

a) For $g \in W$*,* $\bar{g}(\bar{U}_1) \subset U_2$*.*

b) For every $g \in W$*, there are maps* h *arbitrarily close to* g *in* M^V *such that* $\bar{h}|U_1$ *is transverse to* N*.*

Proof. We first choose a C.N. in $J^r(V, M)$ at Q, within which N is given by equations $H_\lambda = 0 (1 \leqq \lambda \leqq p)$, where the H_λ are smooth functions with linearly independent differentials. Hence we can find a subset $\{z_\mu \colon 1 \leqq \mu \leqq p\}$ of the coordinates $x_i, \mu_{\omega,j}$ at Q such that $|\partial H_\lambda / \partial z_\mu| \neq 0$ at Q_0, say without loss of generality it is positive.

Now, having fixed in advance the local coordinate at P and Q, we may take for U_2 any neighbourhood of Q within which N is defined by the equation $H_\lambda = 0$ and $|\partial H_\lambda / \partial z_\mu| > D > 0$. We choose U_1' such that $\bar{f}(\bar{U}_1') \subset U_2$): these will nearly be the neighbourhoods i), ii) of the lemma. It will be convenient to write (without loss of generality)

$$z_\lambda = x_\lambda \quad (1 \leqq \lambda \leqq q) \quad z_\lambda = U_{\omega_\lambda, j_\lambda} \quad (q < \lambda \leqq p).$$

In order to obtain the result, we must now take a map g, with $\bar{g}(\bar{U}_1') \subset U_2$, and attempt to deform g to be transverse to N. We shall define the deformation locally; it may be extended to the rest of the manifold by using bump functions. We define $G \colon V \to M$ by

$$G_j(x_1, \ldots, x_v, \varepsilon_1, \cdots, \varepsilon_p) = g_j(x_1 + \varepsilon_1, \cdots, x_q + \varepsilon_q, x_{q+1}, \cdots, x_p) + \sum_{\substack{q < \lambda \leqq p \\ j_\lambda = j}} c_\lambda \varepsilon_\lambda x^{\omega_\lambda}$$

where the c_λ are constants to be defined. We shall calculate the partial derivatives of the $H_\lambda(\bar{G})$ with respect to the ε_λ at $\varepsilon = 0$. Now

$$\frac{\partial H_\lambda(\bar{G})}{\partial \varepsilon_\mu} = \sum_{\omega, j} \frac{\partial H_\lambda(\bar{G})}{\partial U_{\omega,j}} \frac{\partial U_{\omega,j}}{\partial \varepsilon_\mu} \tag{4.4}$$

But by definition, $U_{\omega,j}(\bar{G}) = \partial_\omega G_j$ so

$$U_{\omega,j}(\bar{G} = \partial_\omega g_j(x_1 + \varepsilon_1, \cdots, x_q + \varepsilon_q, x_{q+1}, \cdots, x_p) + \sum_{\substack{q < \lambda \leqq p \\ j_\lambda = j}} c_\lambda \varepsilon_\lambda \partial_\omega x^{\omega_\lambda}$$

and

$$\frac{\partial U_{\omega,j}(\bar{G})}{\partial \varepsilon_\mu} = \begin{cases} 0 & \text{if } \mu > q, j \neq j_\mu, \\ c_\mu \partial_\omega x^{\omega_\mu} & \text{if } \mu > q, j = j_\mu, \\ \frac{\partial}{\partial x_\mu} U_{\omega,j}\{\bar{G}(x_1 + \varepsilon_1, \cdots, x_q + \varepsilon_q, x_{q+1}, \cdots, x_p)\} & \text{if } \mu \leqq q. \end{cases}$$

Now set $x = \varepsilon = 0$. Then $\partial_\omega x^{\omega_\mu} = 0$ unless $\omega = \omega_\mu$, in which case $\partial_\omega x^{\omega_\mu} = \omega_\mu!$. We set $c_\mu = (\omega_\mu!)^{-1}$. Hence at $x = \varepsilon = 0$,

$$\text{if } \mu \leqq q, \qquad \frac{\partial U_{\omega,j}(\bar{G})}{\partial \varepsilon_\mu} = \frac{\partial U_{\omega,j}(\bar{G})}{\partial x_\mu}$$

$$\text{if } \mu > q, \qquad \frac{\partial U_{\omega,j}(\bar{G})}{\partial \varepsilon_\mu} = \begin{cases} 0 & \text{if } (\omega, j) \neq (\omega_\mu, j_\mu) \\ 1 & \text{if } (\omega, j) = (\omega_\mu, j_\mu) \end{cases}$$

and so substituting in (4.4),

$$\frac{\partial H_\lambda(\bar{G})}{\partial \varepsilon_\mu} = \begin{cases} \sum_{\omega,j} \frac{\partial H_\lambda(\bar{G})}{\partial U_{\omega,j}} \frac{\partial U_{\omega,j}}{\partial x_\mu} = \frac{\partial H_\lambda(\bar{G})}{\partial x_\mu} & (\mu \leqq q), \\ \frac{\partial H_\lambda(\bar{G})}{\partial U_{\omega_\mu, j_\mu}} & (\mu > q). \end{cases}$$

thus in any case,

$$\frac{\partial H_\lambda(\bar{G})}{\partial \varepsilon_\mu} = \frac{\partial H_\lambda(\bar{G})}{\partial z_\mu} \qquad \text{at } x = \varepsilon = 0.$$

We are now ready to complete the proof of the lemma. For any g defined (at least) on a neighbourhood of $\bar{U}_1'$, with $\bar{g}(\bar{U}_1') \subset U_2$, we define

$$K(g, \varepsilon) = |\frac{\partial H_\lambda(\bar{G})}{\partial \varepsilon_\mu}|;$$

this is a function on $\bar{U}_1'$, and we have checked that at P we have $K(f, 0) > D > 0$. Now choose δ, and then U, such that on $\bar{U}_1$, we have $K(f, \varepsilon) > \frac{2}{3}D$, provided $|\varepsilon| \leqq \delta$. Then W is the set of maps g with $\bar{g}(\bar{U}_1) \subset U_2$ and $K(g, \varepsilon) > \frac{1}{3}D$ on $\bar{U}_1$, provided $|\varepsilon| \leqq \delta$: this clearly defines an open set in M^V.

In particular, for $g \in W$, $K(g)$ is nonzero on U_1. By the Implicit Function Theorem (4.3, Part 0), the equations

$$H_\lambda(\bar{G}(x_1, \cdots, x_v, \varepsilon_1, \cdots, \varepsilon_p)) = 0$$

define $\varepsilon_1, \cdots, \varepsilon_p$ as smooth functions of $x_1, \cdots, x_v$, with $|\varepsilon| < \delta$, in an open subset of U_1 (points whose image under $\bar{g}$ are close to N). By Sard's theorem, we can find arbitrarily small regular values ε° of this map. But at a regular value, $d\varepsilon_1, \cdots, d\varepsilon_p$ are linearly independent functions of $dx_1, \cdots, dx_v$; and since $K(g, \varepsilon^\circ)$ is nonzero, $dH_1, \cdots, dH_p$ are linearly independent functions of these. Hence the induced map from $N^\perp \subset J^*$ (which admits the dH_λ as basis) to V^* is monomorphic on U_1 for $\bar{G}(x, \varepsilon^\circ)$, i.e., $\bar{G}(x, \varepsilon^\circ)|U_1$ is transverse to N. Taking ε° small, it also approximates $g(x)$. □

It is now easy to prove the general theorem:

Theorem 4.5. *Let N be a submanifold of $J^r(V, M)$. The set of maps $f \colon V \to M$ transverse to N is dense in M^V.*

Proof. First let K be a compact subset of V. Then K can be covered by a finite number of the neighbourhoods U_1^α of the lemma. The intersection of the corresponding sets W is an open neighbourhood of f, and the subset of W of functions g with $g|U_1^\alpha$ transverse to N is dense, by the lemma. By the Corollary 3.6 (to Baire's theorem), the subset of g with $g|K$ transverse to N is also dense (Baire's theorem applies to an open subset of a complete metric space - see 2.22, Part 0), and is open, being defined by mapping a compact subset of V to an open subset of a jet space.

Since f was arbitrary, we now see that the set of g with $g|K$ transverse to N is a dense open set. The result follows by a second application of the Corollary to Baire's theorem 3.6. □

Complement 4.6. *If V is compact, the set of $f: V \to M$ transverse to N is also open in M^V.*

Proof. This was established in the proof of the above theorem. □

In general, the set of f is a dense G_δ set; by further applications of Baire's theorem, we see that he set of f satisfying a finite, or even countable, number of conditions of the above type is still dense.

We now derive a number of extensions of the above theorem: these are rather more useful than the result in its original form.

Proposition 4.7. *If F is closed in V and $f|F$ is transverse to N, then f can be approximated by g, transverse to N, and with $g|F = f|F$.*

Proof. Consider the subspace of M^V of functions agreeing with f on V. Since, if h is such a function, h is transverse to N above an open neighbourhood of V, we can apply Baire's theorem as in the proof of 4.5 (the space is clearly still complete). □

Proposition 4.8. *Let N be a cell-complex contained in $J^r(V, M)$, with* $\operatorname{codim} N > \dim N$. *Then the set of f with $\bar{f}(V)$ disjoint form N is dense in M^V.*

Proof. We proceed by induction on $\dim N$. Suppose the proposition has been proved for dimension $i-1$. Then any f can be approximated by g with $\bar{g}(V)$ disjoint from the skeleton N^{i-1}. But now any h sufficiently close to g also avoids N^{i-1}, and we can apply the theorem to the manifold $N^i \setminus N^{i-1}$ to make h transverse to (and so avoiding) that. □

Corollary 4.9. *Let $N \subset J^r(V, M)$ have a subcomplex K whose codimension (in J) is $> \dim V$, and with $N \setminus K$ a manifold. The set of f with $\bar{f}(V)$ disjoint from K and transverse to $N \setminus K$ is dense.*

Proof. As for 4.8, any f may be approximated by g avoiding K, and then apply the theorem (taking an approximation close enough still to avoid K). We obtain h, as desired. □

Proposition 4.10. *Let N be a submanifold of $J^r(V_1, M_1) \times J^r(V_2, M_2)$. Then the set of $(f_1, f_2) \in M_1^{V_1} \times M_2^{V_2}$ such that $\bar{f}_1 \times \bar{f}_2$ is transverse to N is dense in $M_1^{V_1} \times M_2^{V_2}$.*

Proof. Follow the proof of Lemma 4.3: we there found variations ε_1 say of f_1, and ε_2 of f_2. Taking these as a simultaneous variation, the remainder of the proof can be completed without essential change. □

Proposition 4.11. *Let N be a submanifold of $J^r(V, M) \times J^r(V, M)$, D an open neighbourhood of the diagonal $\Delta(V)$ in $V \times V$, $C = V \times V \setminus D$. Then the set of $f \in M^V$ such that $(\bar{f} \times \bar{f})|C$ is transverse to $N$is dense in M^V.*

Proof. By 2.3, Part 0, we may cover C by a countable union of products of discs $U_1^\alpha \times U_2^\alpha$ where U_1^α, U_2^α are disjoint. By Proposition 4.10, the set of pairs $f_1 \colon U_1^\alpha \to M$, $f_2 \colon U_2^\alpha \to M$ with $\bar{f}_1 \times \bar{f}_2$ transverse to N is a dense subset. It follows (from a definition of topology on M^V) that the set of $f \colon V \to M$ with $\bar{f}_1|U_1^\alpha \times \bar{f}_2|U_2^\alpha$ transverse to N is a dense open set. The required set is the intersection of all these, so by Baire's theorem (2.21, Part 0) is still dense. □

Corollary 4.12. *Let N be a submanifold of $J^r(V, M) \times J^r(V, M)$, $f \colon V \to M$ such that $(\bar{f} \times \bar{f})(\Delta)$ does not meet N. Then we can approximate f by a map g, transverse to N, and with $(\bar{g} \times \bar{g})(\Delta)$ disjoint from N.*

Proof. Since N is closed , some neighbourhood of $(\bar{f} \times \bar{f})(\Delta)$ also avoids N: we may take the inverse image of a smaller neighbourhood ad D in the above. But for any sufficiently close approximation g to f, $(\bar{g} \times \bar{g})(D)$ is still disjoint from N. □

There are of course numerous results which can be obtained by a judicious combination of these extensions, but it does not seem worth attempting to formulate a common generalisation of them all.

Chapter 5

Applications

Theorem 5.1. *Let M^m be a smooth manifold, N^n a submanifold, V^v a manifold with boundary. Then any $f\colon V \to M$ can be approximated by maps g transverse to N, and if $f|\partial V$ is transverse to N, we may suppose $g|\partial V = f|\partial V$.*

Proof. Apply Proposition 4.8 with $r = 0$, and considering the submanifold $V^v \times N^n$ of $V^v \times M^m = J^0(V, M)$. The last clause follows from Proposition 4.7. □

This was an early form of the transversality theorem, and is useful for applications to cobordism theory.

Theorem 5.2. *Suppose $m \geqq 2r$. Then immersions of V in M are dense in M^V.*

Proof. Consider the subset N of $J^1(V, M)$ consisting of singular jets, i.e., of jest where the matrix (U_{ij}) has rank $< v$. This is defined by the vanishing of $(m - v + 1)$ determinants in general, so is a simplicial complex of codimension at least $m - v + 1 \geqq v + 1$. By Proposition 4.8, the set of maps $f\colon V \to M$ with $\bar{f}(V)$ disjoint from N is dense. But these are just immersions. □

Theorem 5.3. *Suppose $m \geqq 2r + 1$. Then imbeddings of V in M are dense in M^V, provided V is compact. If not, imbeddings as closed submanifolds are dense in the set of proper maps.*

Proof. First suppose V compact: then any $1{-}1$ immersion is an imbedding. Now any $f\colon V \to M$ can be approximated by an immersion g, by Theorem 5.2. Since g is an immersion, for some neighbourhood D_1 of $\Delta(V)$ in $V \times V$, no distinct pair of points in D_1 have a common image under g. We shall now apply Corollary 4.12, taking $D \subset D_1$ and N as the set of pairs of jets in $J^0(V, M) \times J^0(V, M)$ with the same image (i.e., $V \times V \times \Delta(M)$). This has codimension m , so since $m > 2v$, $\bar{h} \times \bar{h}$ is transverse to N on $C = V \times V \times \backslash D$ only if $(\bar{h} \times \bar{h})(C)$ is disjoint from N. But if h approximates closely enough to g, by 4.6, h is still an immersion, and h will not identify pairs of points which lie in D. Then h is $1 - 1$, and so an imbedding.

For V non-compact, we express it as a countable increasing union of compact subsets V_i. By the above, the st of f with $f|V_i$ an imbedding is a dense open set, hence the intersection of all these is still dense. Since the modification on f to be an imbedding n each V_i can be made smaller as we move further out, we may find such an approximation to any proper map which is another one. The result then follows by 2.8, Part I. □

Even this is not the final form of Whitney's theorem - a further argument along the same lines proves

Complement 5.4. *If $m \geqq 2v + 1$, and $f: V \to M$ is proper onto $f(V)$, then f can be approximated by an imbedding.*

We will not go into the details, since the argument really uses a different topology on M^V from that considered above. Now we can similarly improve the results of Chapter 2.

Theorem 5.5. *Non-degenerate functions are dense in $\mathbb{R}^v$.*

Proof. (Cf. 5.2 above). Let N be the subset of singular jets in $J^1(V, \mathbb{R})$: this is given in local coordinates by the equations $u_i = 0$, so is a submanifold. By Theorem 4.5, the set of functions f which are transverse to N is dense.

We now say that $\bar{f}$ is transverse to N if and only if f is non-degenerate. P is a critical point of f when $\bar{f}(P) = Q \in N$. Taking local coordinates as usual at P, Q we must calculate

$$\begin{aligned} d\bar{f}(\frac{\partial}{\partial x_i}) &= \frac{\partial}{\partial x_i} + \frac{\partial f}{\partial x_i}\frac{\partial}{\partial y} + \sum \frac{\partial^2 f}{\partial x_i \partial x_j}\frac{\partial}{\partial u_j} \\ &= \frac{\partial}{\partial x_i} + \sum_j \frac{\partial^2 f}{\partial x_i \partial x_j}\frac{\partial}{\partial u_j} \end{aligned}$$

since at Q, $\partial f/\partial x_i = 0$. But the tangent space to N is spanned by $\partial/\partial y$ and the $\partial/\partial x_i$ (since N is defined by the equation $u_i = 0$), and these with the above span J_Q if and only if the matrix $\partial^2 f/\partial x_i \partial x_j$ is non-singular, i.e., Q is a non-degenerate critical point of f. □

In fact one can make this a little more precise yet. If $\{P_\alpha\}$ are the critical points of f, recall that $\{f(P_\alpha)\}$ are the critical values.

Proposition 5.6. *Non-degenerate functions with all critical values distinct are dense.*

Proof. Let N be the submanifold of $J^1(V, \mathbb{R}) \times J^1(V, \mathbb{R})$ given by pairs of singular jets with the same image (i.e., value). This has codimension $2v + 1$ (as N in Theorem 5.5 has codimension v). By that theorem, any f can be approximated by g with only non-degenerate critical points. Since these are all isolated, there is a neighbourhood D of $\Delta(V)$ in $V \times V$ containing no pair of critical (for g) points; à fortiori, $g(D)$ avoids N. By Corollary 4.12, we can approximate g by a map h transverse to (and so avoiding) N everywhere - of course, h can still be taken non-degenerate. □

Such functions are called *generic.* in general, given v, m, a generic map of V^v to M^n is to be thought of as one which satisfies all the transversality conditions which can be stated in terms of v, m alone (using no special facts about V, M). To find a satisfactory general definition of the word "generic" in this context is still an unsolved problem. The above is the case $m = 1$, and Theorem 5.3 is the case $m \geqq 2v + 1$. We now discuss a very general case, namely when $2m > 3v$; we shall use the results in later, for Haefliger's imbedding theorem.

We need make, in all, six applications of the transversality theorem. First, let N_1 be the subvariety of $J^1(V, M)$ consisting of jets with rank $\leqq v - 2$ (here, we use "*variety*" to denote a submanifold with singularities - for our purpose this may be defined as a countable, finite-dimensional CW-complex). For a $v \times m$ matrix to have rank $v - 2$, imposes some conditions: now in an open subset of the space of such matrices, the first $v-2$ columns are linearly independent, and the condition is then that the remaining $m - v + 2$ lie in a subspace of $\mathbb{R}^v$ of codimension 2. Hence the codimension of this set of matrices, hence of N_1, is $2(m - v + 2)$, which is grater than v if $2m \geqq 3v - 3$, so by Proposition 4.8, the set of f with $\bar{f}(V)$ disjoint from N_1 is a dense G_δ-set.

Next, let N_2 be the subvariety of $J^1(V, M)$ consisting of singular jets (i.e., of tank $\leqq v - 1$). Then by Corollary 4.9, we may suppose f transverse to N_2 (since the singularities of N_2 all lie on N_1. Hence, by Lemma 4.2, $\bar{f}^{-1}(N_2)$ is a submanifold of V, whose codimension is that of N_2, namely $(m - v + 1)$. We call this the *singular manifold* Σ of f: at each point of Σ, *df* has rank $(v - 1)$. The dimension of Σ is $(2v - m - 1)$.

Now let N_3 be the subvariety of $J^2(V, M)$ consisting of singular jets of rank $v-1$ of a function at P such that $\ker(df)_P \subset \Sigma_P$, and jest of rank $\leqq v-2$. Since Σ has codimension $m - v + 1$, the condition $\ker(df)_P \subset \Sigma_P$ imposes $m - v + 1$ further conditions, and N_3 has codimension $2(m - v + 1)$. By Proposition 4.8, provides this exceeds v i.e., $2m \geqq 3v - 1$, we may suppose that $\bar{f}(V)$ avoids N_3. Observe that this means that at each point of Σ, $d\bar{f}(\ker df)$ is not tangent to N_2. We now phrase these three normalisations in terms of analysis. Firs take coordinates in V and M, and the usual coordinates in the jet space $J^1(V, M)$. Then we have

$$d\bar{f}(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_i} + \sum_j u_{,j} \frac{\partial}{\partial y_j} + \sum_{j,k} \frac{\partial^2 y_j}{\partial x_i \partial x_k} \frac{\partial}{\partial u_{kj}}$$

where, we recall $u_{ij} = \partial y_j / \partial x_i$. Now by the firs normalisation, at each critical point P, *df* has rank $v - 1$. We suppose coordinates chosen so that at P, $\partial / \partial x_i$ spans $\ker(\partial f)$, thus at P

$$0 = df \frac{\partial}{\partial x_1} = \sum_j u_{ij} \frac{\partial}{\partial y_j} \quad \text{i.e.,} \quad 0 = u_{ij} \quad (1 \leqq j \leqq m).$$

Then at $Q = \bar{f}(P)$, N_2 may be locally described as the set of jets such that the first row of (u_{ij}) is a linear combination of the rest, i.e., for suitable ε_i,

$u_{ij} = \sum_2^v \varepsilon_i u_{ij}$ for all j. Hence the tangent space to N_2 at Q has as basis the

$$\frac{\partial}{\partial x_i}, \quad \frac{\partial}{\partial y_j}, \quad \frac{\partial}{\partial u_{ij}} \quad (i \neq 1) \quad \text{and} \quad \frac{\partial}{\partial \varepsilon_i}, \quad \text{where} \quad \frac{\partial}{\partial \varepsilon_i} = \sum_j u_{ij} \frac{\partial}{\partial u_{ij}}.$$

Now the condition that f is transverse to N_2, i.e., $d\bar{f}(V_P) + (N_2)_Q = J_Q$ states that the space spanned by the $\partial^2/\partial u_{1j}$ is also spanned by the $\sum_j u_{ij}\partial/\partial u_{1j}$ (from $(N_2)_Q$), and the $\sum_j \partial^2 y_j/\partial x_1 \partial x_k \partial/\partial u_{1j}$ (from $d\bar{f}(V_P)$). Also, the condition that f is transverse to N_3 i.e., $d\bar{f}(\partial/\partial x_1)$ is not tangent to N_2, now states that the first of the last set of vectors is linearly independent of the first set (they are linearly independent of each other since df has rank $v-1$).

To simplify this, first choose the coordinates in V so that the $\partial/\partial x_i$ where $(m - v + 2 \leqq i \leqq v)$ span Σ_P: then the $d\bar{f}(\partial/\partial x_i)$ corresponding lie in $(N_2)_Q$. The matrix whose rows are

$$\frac{\partial^2 y_j}{\partial x_1^2}, \quad \frac{\partial y_j}{\partial x_i} \quad 2 \leqq i \leqq v \quad \text{and} \quad \frac{\partial^2 y_j}{\partial x_1 \partial x_i} \quad 2 \leqq i \leqq m - v + 1$$

is now non-singular: we make a linear transformation of the y_j to reduce it to the unit matrix. Then by Taylor's theorem,

$$y_1 = \frac{1}{2}x_1^2 + Q_1(x_2, \cdots, x_v) + \cdots$$

for $2 \leqq j \leqq v$,

$$y_j = x_j + Q_j(x_1, \cdots, x_v) + \cdots$$

and for $2 \leqq i \leqq m - v + 1$,

$$y_{i+v-1} = x_1 x_i + Q_{i+v-1}(x_2, \cdots, x_v) + \cdots$$

where the Q_j are quadratic, and dots represent terms of higher order. Finally, put

$$\begin{aligned} x'_j &= x_j + Q_j(x_1, \cdots, x_v); \\ y'_1 &= y_1 - Q_1(y_2, \cdots, y_v), \quad \text{and} \\ y'_{i+v-1} &= y_{i+v-1} - Q_{i+v-1}(y_2, \cdots, y_v) \end{aligned}$$

- clearly all allowable changes - and the quadratic terms drop out too, so that modulus terms of the third and higher orders, f is described in a neighbourhood of P by

$$y_1 = \frac{1}{2}x_1^2, \quad y_j = x_j, \quad y_{i+v-1} = x_1 xi. \tag{5.7}$$

We shall see later that by further coordinate transformations, f may be seen to take exactly this form.

Our further normalisations are concerned with double points, rather than singular points, of f. Next let N_4 be the subvariety of $J^1(V, M) \times J^1(V, M)$ consisting of pairs if jets with the same image, one of which (say the first) is singular. We wish to apply Corollary 4.12. now certainly in a neighbourhood of $(P, P) \in V \times V$, the image of $\bar{f}$ avoids N_4 if P is not a singular point. Suppose then $P \in \Sigma$. Then in a neighbourhood of P, the function f is described by th equations above. If $f(0, 0, \cdots, 0) = f(x_1, x_2, \cdots, x_v)$ for small x_i, then to the second order in them, $x_j = 0$ $2 \leqq j \leqq v)$, equating the corresponding y_j, and $\frac{1}{2}x_1^2 = 0$, equating the corresponding y_1. Hence all the x_i vanish. So in some neighbourhood of (P, P), $\bar{f} \times \bar{f}$ does avoid N_4 (except on $\Delta(\Sigma)$). Since N_4 has codimension $m + (m - v + 1)$, greater than $2v$ if $2m \geqq 3v$, by Corollary 4.12, we can approximate f by a map (let us again call it f) such that $\bar{f}$ avoids N_4.

Now let N_5 be the subvariety of $J^0 \times J^0$ consisting of pairs of jets with the same image. Again, before we apply the theorem, we must investigate the neighbourhood of a critical point P. We shall use equations (5.7) as exact - the error will always be small; and we suppose x_1 not of a smaller order of magnitude than the other x_i (otherwise, refer coordinates to a different point P' on Σ). Then clearly two points $(x_1, \cdots, x_{v)}$ $(x'_1, \cdots, x'_v)$ have the same image only if

$$x'_1 = \pm x_1, \quad x'_j = x_j \quad (2 \leqq j \leqq m), \quad x'_1 x'_i (= x'_1 x_i) = x_1 x_i (2 \leqq i \leqq m - v + 1)$$

so for distinct points,

$$x'_1 = -x_1, \quad \text{and} \quad x_i = x'_i = 0 \quad (2 \leqq i \leqq m{-}v{+}1); \quad x_i = x_i \quad (m{-}v{+}2 \leqq i \leqq v).$$

Now we have

$$df \frac{\partial}{\partial x_1} = x_1 \frac{\partial}{\partial y_1} + \sum_{2}^{m-v+1} x_i \frac{\partial}{\partial y_{i+v-1}};$$

for $2 \leqq i \leqq m - v + 1$,

$$df \frac{\partial}{\partial x_1} = \frac{\partial}{\partial y_1} + x_1 \frac{\partial}{\partial y_{i+v-1}}$$

and for $m - v + 2 \leqq i \leqq v$,

$$df \frac{\partial}{\partial x_1} = \frac{\partial}{\partial y_1}$$

and for the other point, change the sign of x_1. Then since $x_1 \not\equiv 0$, it is clear that these vectors span the tangent space to M at the common image of the two points.

Now to say that $\bar{f} \times \bar{f}$ is transverse to N_5 is the same as to say that when $f(P) = f(P') = Q$, then $df(V_P) + df(V_{P'}) = M_Q$. We check this near the diagonal: if two adjacent points have a common image, they are adjacent to a critical point, and we have just checked the condition in the neighbourhood of a critical point. Hence we can apply Corollary 4.12, and suppose $\bar{f} \times \bar{f}$ transverse to N_5 (except on ΔV, where the condition does not make sense). Thus the inverse image of N_5 in $V \times V$ is a submanifold: this is not tangent to (say) the first vector V (except at a critical point), so its projection in the second factor V is an immersion. The image is the set of double points Δ of f.

Finally let N_6 be the subvariety of $J^0 \times J^0 \times J^0$ consisting of triples of jets with the same image. We can apply Corollary 4.12 strengthened for triples (instead of pairs). Firs check that three points of V, of which two are neighbouring, cannot have the same image under f; now the neighbouring ones must be near a critical point P, and a point distant from P has a distant image (by the fourth step), and from the details above, we see that three points close to P cannot have a common image. Hence we can make f transverse to N_6; since this has codimension $2m$, we can avoid triple points if $2m > 3v$.

Theorem 5.8. *Let M^m, V^v be smooth manifolds, $2m > 3v$. then any $g\colon V \to M$ may be approximated by an f, which is an imbedding except as follows. There are double points, forming a submanifold Δ of dimension $2v - m$, and singular points, forming a submanifold Σ of codimension 1 in Δ. Near Σ, f is given locally by (5.7). Hence $D = f(\Delta)$ is a submanifold of M with boundary $S = f(\Sigma)$*

Proof. We have seen that Δ is an immersed submanifold; when there are no triple points it is imbedded. That Δ remains a manifold near Σ, with Σ as submanifold, follows from the equations above: Δ is simply given by $x_i = 0$ $(2 \leqq i \leqq m - v + 1)$ (modulo higher terms). Moreover $f(\Delta)$ is also clearly a submanifold, except perhaps near $f(\Sigma)$; but there it is locally given by $y_1 \geqq 0$, $y_i = 0$ $(2 \leqq j \leqq m - v + 1$ and $v + 1 \leqq j \leqq m)$ which makes the matter quite clear. □

Part III

Immersions and Imbeddings

This part has not been written.

Part IV

Theory of Handle Decompositions

These notes are a continuation of

- Part 0 Analytical Foundations
- Part I Geometrical Foundations
- Part II Theorems of Transversality and General Position

issued in Cambridge in 1962 (copies available on request from the Department of Pure Mathematics). They are based on:

- lectures given in
  - Oxford (October - December, 1962) and
  - Cambridge (January - March, 1964),
- and various seminars held in Cambridge.

Thanks are due to:

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It is intended that further pasts shall be as follows:

- Part III Immersions and Imbeddings - a few enigmatic references to III are needed in IV -
- Part V Cobordism, and
- Part VI Surgery;

these will probably appear at about yearly intervals (though I hope sooner). Suggestions for improvements in presentation will be welcome, in anticipation of attempts to rewrite the notes more comprehensibly.

Chapter 1

Existence

Definition 1.1. Let W be a manifold, and suppose $\partial_- W$ and $\partial_+ W$ disjoint manifolds with union ∂W. Then the pair $(W, \partial_- W)$ is a *cobordism*. We call the pair $(W, \partial_+ W)$ the *dual cobordism*. We also call W a cobordism of $\partial_- W$ to $\partial_+ W$, and say that $\partial_- W$, $\partial_+ W$ are *cobordant*. If W is a manifold with corner, and $\partial_- W$, $\partial_c W$, $\partial_+ W$ are parts of the boundary such that $\partial_- W$ are $\partial_+ W$ disjoint, $\partial \partial_c W = \angle W = \partial(\partial_- W \cup \partial_+ W)$, we call W a *cobordism with corner*. We shall usually denote a cobordism by a single letter and often just call it a manifold. For example, we usually regard a product $M \times I$ as a cobordism, with $\partial_-(M \times I) = M \times 0$, $\partial_+(M \times I) = M \times 1$; if M has boundary, write $\partial_c(M \times I) = \partial M \times I$. Our manifolds will be compact unless otherwise stated.

Suppose M^m a cobordism, $f\colon S^{r-1} \times D^{m-r} \to \partial_+ M$ an imbedding. Introduce a corner (I, 6.10) along $f(S^{r-1} \times S^{m-r-1})$. Now glue $D^r \times D^{m-r}$ to M by f (I, 7). We know this gives a result unique up to diffeomorphism. This is described as M *with an r-handle attached by* f, or as $M \cup_f h^r$, and f as the *attaching map* of the handle. We call r the *dimension* of the handle. We define $\partial_+(M \cup h^r) = (M \setminus \mathrm{Im} f) \cup (D^r \times S^{m-r-1})$. If we have a sequence of attached handles:

$$N = M \cup_{f_1} h^{r_1} \cup \cdots \cup_{f_k} h^{r_k},$$

we describe this as a *handle presentation* of N on M; if the maps f_i are not specified, as a *handle decomposition*. In particular, if $M = Q \times I$, we speak of a handle decomposition of N with *base* Q (here, Q may be empty). Observe the similarity of this definition to that of a CW complex: one of our main objects will be to show how the theory parallels that of finite CW complexes. The purpose of this chapter is to prove the existence of handle decompositions for compact manifolds: in the next few chapters we will show how to reduce such a decomposition (under some hypotheses) to its simplest form.

To prove existence, we shall use non-degenerate functions.

Lemma 1.2. *Any cobordism W admits a non-degenerate function f, with all critical values distinct, attaining an absolute minimum on $\partial_- W$ only, and an absolute maximum on $\partial_+ W$ only.*

Proof. Let $\partial_- W \times I$, $\partial_+ W \times I$ be tubular neighbourhoods of $\partial_- W, \partial_+ W$ which are disjoint (Lemma I, 3.8). Define $g\colon W \to [-1, 2]$ by:

$$g(x,t) = \begin{cases} t-1 & \text{for } x \in \partial_- W, \\ 2-t & \text{for } x \in \partial_+ W. \end{cases} \tag{1.3}$$

and some extension to be a continuous function taking only values between 0 and 1 elsewhere: this is possible since W is normal. Approximate g by a smooth function h, agreeing with g near ∂W (use a partition of unity, as in 0, 2.2). Now approximate h by a non-degenerate function f with distinct critical values (II, 4.10) agreeing with h, and so g, near ∂W - which is possible (II, 4.7) since g and h have no critical points in a neighbourhood of ∂W. □

Complement 1.4. *We may suppose that for x close to ∂W, f is defined by the formula (1.3).*

Now we give W a Riemannian structure (0, 3.8) adapted to the boundary (Definition I, 3.9); for convenience we suppose it as in (I, 3.11) - that is, a product metric in some neighbourhood of ∂W. Then the differential 1-form *df* induces at each $P \in W$ an element df_P of W_P^*; using the Riemannian structure, this is identified with an element of W_P - i.e., a tangent vector. Thus *df* gives a vector field, which we call ∇f.

In $\mathring{W}$, we can use (0, 4.7) to integrate f and obtain $\varphi_t(P)$, each defined for a certain range of values of P. Near a point of $\partial_- W$, we can take coordinates $x_1, \ldots, x_n$ such that W is defined by $x_1 \geqq 0$, x_1 is the t-coordinate in the tubular neighbourhood, so that $f(x) = x_1 - 1$ and the Riemannian structure is of the form $ds^2 = dx_1^2 + \sum_{i,j=2}^n g_{ij} dx_i dx_j$. Hence ∇f agrees with $\partial/\partial x_1$ in such a neighbourhood, and orbits are of the form

$$\varphi_t(x_1, \ldots, x_n) = (x_1 + t, x_2, \ldots, x_n) \qquad x_1 \geqq 0, x_1 + t \geqq 0.$$

Each of them meets $\partial_- W$ in just one point, and together they fill out a neighbourhood of $\partial_- W$ in a 1-1 manner. Similarly for $\partial_+ W$.

If we regard $\varphi_t(P)$ as a function of t, it is smooth, and we have a metric, so can speak of speed.

Lemma 1.5. *(a)* $\frac{d}{dt} f(\varphi_t(P))|_{t=0} = |df_P|^2$

(b) The speed of $\varphi_t(P)$ at $t = 0$ *is* $|df_P|$.

Proof.

$$\begin{aligned} (a) \frac{d}{dt} f(\varphi_t(P))|_{t=0} &= \nabla f(f)|_P \quad \text{by definition of} \quad \varphi \\ &= df(\nabla f)|_P \\ &= < df_P, df_P > = |df_P|^2 \end{aligned}$$

in the Riemannian inner product on W_P, since this defined ∇f.

(b) Take coordinates $(x_1, \dots, x_m)$ at P (so that P has coordinates $(0, \dots, 0)$) such that at P the Riemannian metric agrees with the standard metric in $\mathbb{R}^n$. Let $df = \sum a_i dx_i$: then

$$\nabla f = \sum a_i \partial/\partial x_i \qquad \text{(at } P\text{)}$$

Thus, at P.

$$\frac{\partial \varphi_t(P)}{\partial x_i} = a_i$$

so the speed of $\varphi_t(P)$ is just $(\sum a_i^2)^{1/2} = |df_P|$. □

Now suppose $P \in \mathring{W}$, and that the maximum range of t in which $\varphi_t(P)$ is defined is (a, b).

Lemma 1.6. *Suppose W is compact. Then either a is finite and as $t \to a$, $\varphi_t(P)$ tends to a point on $\partial_- W$, or $a = -\infty$, and the closure of each $\{(\varphi_t : t \leqq -K)\}$ contains a critical point of f. (Similarly for b).*

Proof. If a is finite, by Lemma 1.5 (b), the points $\varphi_t(P)$ form a Cauchy sequence as $t \to a$ (since W is compact, $|df_P|$ is bounded); since W is complete, they tend to a limit point Q. If Q was interior to W, it would follow that Q was on the orbit, which could then be extended: thus Q is on ∂W. Since by Lemma 1.5 (a), f increases along each orbit, $f(Q) < f(P)$, so Q is on $\partial_- W$.

Now let $a = -\infty$. Then by Lemma 1.5 (a),

$$\int_{-\infty}^{0} |df_{\varphi_t(P)}|^2 dt$$

converges. So $|df_{\varphi_t(P)}|$ has infimum zero as $t \to -\infty$. Outside any open neighbourhood of the set of critical points, $|df|$ is nonzero, and attains its lower bound (by compactness), so $\varphi_t(P)$ meets any such neighbourhood. But the set of critical points is compact, and so meets the closure of the orbit. □

We are now ready to analyse the function f. For $a \in \mathbb{R}$, we write

$$W^a = \{P \in W : f(P) \leqq a\}$$
$$M^a = \{P \in W : f(P) = a\}$$

thus for

$a < -1$	$W^a = \varnothing$	$M^a = \varnothing$
$a = -1$	$W^a = \partial_- W$	$M^a = \partial_- W$
$a = \varepsilon - 1$	$W^a = \partial_- W \times [0, \varepsilon]$	$M^a = \partial_- W \times \varepsilon$
$a = 2 - \varepsilon$	$W^a = W \setminus \partial_+ W \times [0, \varepsilon]$	$M^a = \partial_+ W \times \varepsilon$
$a = 2$	$W^a = W$	$M^a = \partial_+ W$
$a > 2$	$W^a = W$	$M^a = \varnothing$

provided that ε is so small that $\partial_\eta W \times [0, \varepsilon]$ ($\eta = +, -$) are contained in the neighbourhoods described earlier. Clearly, for $a < b$, $W^a \subset W^b$; we want to describe how W^b is formed from W^a.

Theorem 1.7. *Suppose that for $a \leqq c \leqq b$, c is not a critical value of f. Then*

(a) $f^{-1}[a, b]$ is diffeomorphic to $M^a \times [a, b]$,

(b) W^b is diffeomorphic to W^a.

Remark 1.8. Since a, b are not critical values , M^a, M^b and $f^{-1}[a, b]$ are submanifolds by (II, 4.2).

Proof. (a) Let $a \leqq f(P) \leqq b$. The orbit through P must terminate (at the lower end) at a critical point or at $\partial_- W$, by Lemma 1.6. In either case it meets M^a, for we have assumed the absence of critical points in between. Similarly it meets M^b. Since f increases along orbits, the orbit meets M^a and M^b in just one point each.

Define a map $h\colon f^{-1}[a, b] \to M^a \times [a, b]$ as follows. If $a \leqq f(P) \leqq b$, the first component of $h(P)$ is the unique point where the orbit through P meets M^a. The second component is $f(P)$. h is 1 - 1, for if $h(P) = h(Q)$, then P and Q lie on the same orbit, and have the same value of f; since f increases strictly along orbits, $P = Q$. Also h is onto, for if $R \in M^a$, we know that the orbit through R meets M^b, so if $a \leqq t \leqq b$ there is one (and only one) point P on the orbit with $f(P) = t$, and so $h(P) = (R, t)$. Further, h is smooth, for if $h(P) = (\varphi_{+t}(P))$, $f(P)$, f is smooth, and $\varphi_{-t}(P)$ a smooth function of t and P (0, 4.7) and since $\frac{df(\varphi_t(P))}{dt}$ is nonzero on the orbit, t is a smooth function of P, $f(\varphi_t(P))$, and $f = 0$ defines t as a smooth function of P. Finally h^{-1} is smooth by a similar argument.
(b) It follows from (a) that W^b is obtained from W^a by glueing on $M^a \times I$ along M^a. The result now follows easily: using a tubular neighbourhood of M^a in W^a and the bump function, we could produce an explicit diffeomorphism, and even a weak diffeotopy of it with the identity map of M^b. □

Complement 1.9. *If V is a compact submanifold of W, containing no critical point, and with ∇f nowhere tangent to ∂V, and $\partial_- V$ is the set of points of ∂V at which ∇f points into V, then $V \cong \partial_- V \times I$.*

The proof needs only inessential changes.

The above shows that "as long as a does not pass through a critical value, the diffeomorphism type of W^a remains constant". We now have to investigate the critical value.

Lemma 1.10 (Morse Lemma)**.** *Let f be a smooth function on a neighbourhood of 0 in $\mathbb{R}^n$ with Taylor expansion*

$$f(x) = -\sum_{1}^{\lambda} x_i^2 + \sum_{\lambda+1}^{n} x_j^2 + O(|x|^3).$$

Then there is a smooth coordinate change $y = y(x)$ such that $y(0) = 0$,

$$\frac{\partial y}{\partial x}|_0 = I_n, \quad \textit{and near} \quad 0 \quad f(x) = -\sum_1^{\lambda} y_i^2 + \sum_{\lambda+1}^{n} y_j^2.$$

Proof. We have $f(0) = 0$, so by (0, 3.3) there exist near 0 smooth functions f_i with $f(x) = \sum x_i f_i(x)$. Also, $f_i(0) = \frac{\partial f}{\partial x_i}|_0$, so we can apply the result again to obtain h_{ij} with $f_i(x) = \sum x_j h_{ij}(x)$. Write $g_{ij}(x) = \frac{1}{2}(h_{ij}(x) + h_{ji}(x))$. We think of $f(x) = \sum_{ij} g_{ij}(x) x_i x_j$ as a quadratic form, and diagonalise. Note that

$$g_{ij}(0) = \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}|_0 = \begin{cases} 0 & i \neq j \\ -1 & i = j \leqq \lambda \\ 1 & i = j > \lambda, \end{cases}$$

Set $y_1 = (\pm g_{11}(x))^{-1/2}(\sum_{j=1}^n g_{1j} x_j)$, where the sign is that of $g_{11}(0)$. Then

$$\frac{\partial y_1}{\partial x_1} = \pm 1, \frac{\partial y_1}{\partial x_i} = 0 \quad \text{if } i > 1, \quad \text{and} \quad f(x) = \pm y_1^2 + \sum_{i,j=2}^{n} g'_{ij}(x) x_i x_j.$$

We now repeat the reduction, observing only that although $g'_{ij}(x)$ depends on x_1 we can express x_1 by y_1, and the dependence is smooth. Eventually we obtain the required result. □

Theorem 1.11. *Suppose that for $a \leqq f(P) \leqq b$ there is just one critical point 0, which is non-degenerate and with $f(0) = c$. Then W^b is diffeomorphic to W^a with a handle attached.*

Proof. Our discussion of orbits in Theorem 1.7 remains valid except for those orbits with 0 as a limit point. We must therefore investigate a neighbourhood of 0. Take coordinates $y_1, \ldots, y_n$, with 0 as origin: then in a neighbourhood of 0 we can expand

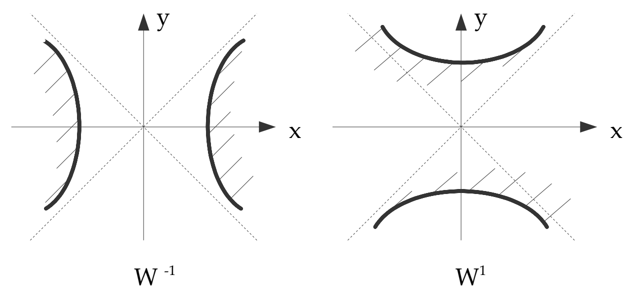
$$f(x) = c + \sum a_{ij} y_i y_j + O(|y|^3) \qquad (a_{ij} = a_{ji}).$$

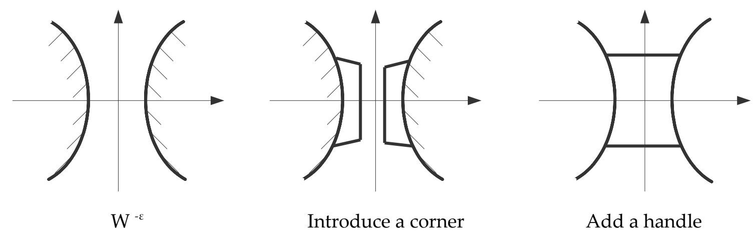
Here (a_{ij}) is the matrix of the Hessian of f at 0; by assumption, this is non-singular. Making an appropriate change of coordinates, we can diagonalise this quadratic form, and write

$$f(x) = c - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2 + O(|x|^3).$$

The integer λ is called the *index* of the Hessian, of f of the critical point 0. By the Morse lemma, we may suppose that the term $O(|x|^3)$ is absent. It will be convenient to suppose that the Riemannian structure agrees with the Euclidean structure in this coordinate system: it is certainly possible to find such a metric.

We draw figures for $f(x) = c - x_1^2 + x_2^2$, showing W^{-1} and W^1. In this case the curves M^ε are hyperbolae with asymptotes $y^2 = x^2$, except for N^0 which





is this line-pair, and as a increases up to zero, W^a increases without essential change, but it engulfs the origin when $a = 0$.

Choose ε small so that $|x| + |y| \leqq 5\varepsilon$, the above formulae are valid. Now consider the following modifications: This show how to imitate W^ε. Formally, write $x = (\xi, \eta)$, where $\xi = (x_1, \ldots, x_\lambda)$, $\eta = (x_{\lambda+1}, \ldots, x_n)$, $f(x) = c - |\xi|^2 + |\eta|^2$, and consider the tube $|\eta| \leqq \varepsilon$, $|\xi| \leqq \varepsilon$: this is the handle. Let V be a smooth manifold with corner which

(i) Coincides with $|\xi| \leqq \varepsilon$, $|\eta| \geqq \varepsilon$ near $|\xi| = \varepsilon$. This includes the corner $|\xi| = |\eta| = \varepsilon$.

(ii) Coincides with $W^{-\varepsilon}$ when $|x| + |y| \geqq 5\varepsilon$, and contains $W^{-\varepsilon}$.

(iii) Has ∂V everywhere transverse to the orbits.

This may be found using a bump function. Then by (I, 6.7) $M^{-\varepsilon}$ is obtained from V by straightening the corner - or equivalently, (I, 6.10), V from $M^{-\varepsilon}$ by introducing one. Now $|\xi| \leqq \varepsilon$, $|\eta| \leqq \varepsilon$, defines a product $D^\lambda \times D^{n-\lambda}$, which meets V in the set $|\xi| = \varepsilon$, $|\eta| \leqq \varepsilon$, an $S^{\lambda-1} \times D^{n-\lambda}$ on their common boundary. Since the union U is evidently smooth, and V and $D^\lambda \times D^{n-\lambda}$ are defined by cutting it along $S^{\lambda-1} \times D^{n-\lambda}$, by (I, 7.4) (in the extended form), U is obtained by glueing these. Now we observe that U is a smooth manifold, transverse to the orbits, with no critical points between it and M^b; thus by complement

1.9, we find W^b diffeomorphic to U. But U consists of W^a with a λ-handle attached. □

Complement 1.12. *If the Hessian of f at c has index λ, we attach a λ-handle.*

Complement 1.13. *If there are several non-degenerate critical points at level c, we attach several handles. Indeed, we can apply the above argument in a neighbourhood of each.*

Corollary 1.14. *W has a handle decomposition on $\partial_- W$,*

It is also possible to proceed the opposite direction.

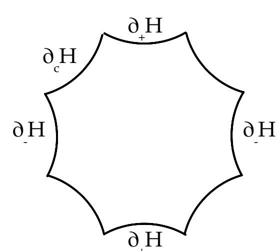
Theorem 1.15. *Given a handle decomposition of W on $\partial_- W$, there is a non-degenerate function f on W (as in Lemma1.2) with just one critical point of index λ for each λ-handle.*

Proof. The result is proved by induction on the number of handles: if there are none, $W \cong \partial_- W \times I$, and we take f as the projection on I. Now let V be defined by attaching all but the last handle: by the induction hypothesis, f can be defined on V, constant on $\partial_+ V$. So if we can define f on $(\partial_+ V \times I) \cup h^\lambda$ we can glue back (using collar neighbourhoods of $\partial_+ V$ on which f reduces to a projection) to make f smooth. Hence we may suppose that W has only one handle.

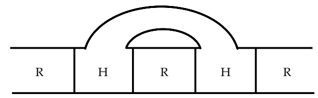
Now let $g\colon S^{\lambda-1} \times D^{n-\lambda} \to \partial_- W$ be the attaching map of a λ-handle. Let R be the closure of the complement of the image. Consider the set $H \in \mathbb{R}^\lambda \times \mathbb{R}^{n-\lambda}$ defined by

$$-1 \leqq -|x|^2 + |y|^2 \leqq 1, |x|^2|y|^2 \leqq 2;$$

define $\partial_- H$, $\partial_+ H$ by $-|x|^2+|y|^2 = -1$, $-|x|^2+|y|^2 = 1$, and $\partial_c H$ by $|x|^2|y|^2 = 2$. Define $G_0\colon S^\lambda \times D^{n-\lambda} \to \partial_- H$ by $G(u,v) = ((1+|v|^2)^{1/2}u, v)$: this is easily



seen to be a diffeomorphism. Define $F\colon H \to [-1,1]$ by $F(x,y) = -|x|^2 + |y|^2$. Now attach H to $R \times [-1,1]$ to form W' by $G\colon \partial_c H \to S^{\lambda-1} \times S^{n-\lambda} \times [-1,1]$, where $G(x,y) = (\frac{x}{|x|}, \frac{y}{|y|}, F(x,y))$. Define $f\colon W' \to [-1,1]$ by $f|H = F$, $f|R =$ projection. This is clearly a smooth function, whose only critical point is the non-degenerate one in H. $\partial_- W'$ is diffeomorphic to $\partial_- W$: take the identity on $R \times -1$, and extend by $g \circ G_0^{-1}$ on $\partial_- H$. Finally, this process gives a manifold diffeomorphic to that obtained by attaching a handle by g; if we use



the construction of Theorem 1.11, it suffices (by a remark in the next chapter) to observe that we have the same attaching sphere and normal framing. □

Chapter 2

Normalisation

We could now proceed immediately to make various deductions about smooth manifolds from the existence of a handle decomposition. First, however, it is convenient to normalise a presentation. Recall that $M \cup_f h^r$ is defined by attaching $D^r \times D^{m-r}$ to M using an imbedding

$$f\colon S^{r-1} \times D^{m-r} \to \partial_+ M.$$

It follows at once from the diffeotopy extension theorem that this is determined up to diffeomorphism by the diffeotopy class of f, for if g is a diffeomorphism of M, g induces a diffeomorphism of $M \cup_f h^r$ with $M \cup_{gf} h^r$. By the tubular neighbourhood theorem, it is even determined by the diffeotopy class of $\bar{f} = f|S^{r-1} \times 0$ together with a homotopy class of normal framing of $f(S^{r-1} \times 0)$ in $\partial_+ M$.

Definition 2.1. Let $M \cup_f h^r$ be a manifold with handle. The *attaching sphere* (or a-sphere) of h^r is the sphere $f(S^{r-1} \times 0)$ in $\partial_+ M$. The *belt sphere* (or b-sphere) is the sphere $0 \times S^{m-r-1}$ in $\partial_+(M \cup_f h^r)$. The *core* is the disc $D^r \times 0$.

Lemma 2.2. *Let $r \leqq s$. Then $(M \cup_f h^s) \cup_g h^r$ may be obtained from M by attaching the handles simultaneously, or in the reverse order.*

Proof. Let $m = \dim M$, $Q = \partial_+(M \cup_f h^s)$. Then we have in Q the a-sphere S^{r-1} of h^r and the b-sphere S^{m-s-1} of h^s. Since $(r-1)+(m-s-1) = m-1-(s+1-r) < m-1 = \dim Q$, by (II, 5.1), S^{r-1} may be approximated by a sphere not meeting S^{m-s-1}: if the approximation is C^1 close enough, we still have an imbedded sphere, diffeomorphic to the old one. By further diffeotopies, we may make S^{r-1} avoid the tubular neighbourhood $D^s \times S^{m-s-1}$ (using the diffeotopy extension theorem, and the obvious fact that the tubular neighbourhood may be 'shrunk' to avoid S^{r-1}) and shrink the tubular neighbourhood $S^{r-1} \times D^{m-r}$ so that, this, too, avoids $D^s \times S^{m-s-1}$. But now the attaching map of the r-handle is disjoint form the s-handle: its image lies in ∂M, and the handles may clearly be added in either order. ∎

Corollary 2.3. *Any W has a handle decomposition on $\partial_- W$ with the handles arranged in increasing order of dimension.*

Proof. Follows at once by induction. □

From now on we shall generally assume that handles have been arranged in order of increasing dimension. Next, we consider handles of consecutive dimensions. To clarify the exposition we describe only the case $W^{n+1} \cup_f h^r \cup_g h^{r+1}$: write M^m for $\partial_+(W \cup_f h^r)$. In M^m we have the a-sphere S^r of h^{r+1} and the b-sphere of h^r. These have complementary dimensions.

By (II, 5.1) the imbedding of S^r may be approximated by a map transverse to S^{m-r}; if the approximation is close enough, we have merely altered the imbedding by a diffeotopy. Now since the dimensions are complementary, and the map transverse, intersections are isolated points; since S^r is compact, there are only finitely many.

We now take an intersection P of S^r with S^{m-r} and normalise $S^r \times D^{m-r}$ in the neighbourhood $D^r \times S^{m-r}$ of P in M. Regard P as in S^{m-r}, so the point is $0 \times P$. First we will deform part of S^r near P to lie along $D^r \times P$; indeed, by the Implicit Function Theorem (0,4.3), the projection of $f(S^r)$ in $D^r \times S^{m-r}$ to $D^r \times P$ is locally a diffeomorphism at $0 \times P$, and there is an obvious diffeotopy along great circle in S^{m-r}. It is easy to extend this diffeotopy to S^r, without introducing any new intersections with S^{m-r}. Next we observe that the tubular neighbourhood $D^r \times S^{m-r}$ of $0 \times S^{m-r}$ can be shrunk, by a diffeotopy, to a smaller concentric tube $D^r \times S^{m-r}$ intersecting S^r in a subset of the $D^r \times P$ above. We can extend this diffeotopy (by the Diffeotopy Extension Theorem) to one of M, and, if we prefer, apply the inverse diffeotopy of $f(S^r)$; this has the effect of stretching out the part where S^r lies along $D^r \times P$ to the whole D^r. Finally, choose a tubular neighbourhood T of P in S^{m-r}; then $D^r \times T$ and $D^r \times D^{m-r}$ are two tubular neighbourhoods of D^r in M, and so diffeotopic (by the Tubular Neighbourhood Theorem); use a diffeotopy to move the imbedding of $S^r \times D^{m-r}$ so that they coincide.

The same process can be done for any intersection other than P.

Definition 2.4. In $W^{m+1} \cup_f h^r \cup_g h^{r+1}$, the handles are in *normal position* if all intersections of $D^r \times S^{m-r}$ and $S^r \times D^{m-r}$ are of the form $D_i^r \times D_i^{m-r}$, where $D_i^r \times D_i^{m-r} \to D^r \times S^{m-r}$ is given by the identity on the first factor and an imbedding (those for separate values of i disjoint) on the second factor, and taking the product; and similarly for $D_i^r \times D_i^{m-r} \to S^r \times D^{m-r}$.

Then the argument above proves

Theorem 2.5. *Any handle presentation of $(W, \partial_- W)$ may be modified by diffeotopies so that*

(i) The handles are arranged in increasing order of dimension,

(ii) Any tow handles of consecutive dimensions are in normal position.

Chapter 3

The homology and homotopy of bundles

It follows from the definition that there is a deformation retraction of $M \cup_f h^r = M \cup_f (D^r \times D^{n-r})$ on $M \cup_f (D^r \times 0)$, so that up to homotopy, attaching a handle is the same as attaching a cell (its core). In fact, it is clear that, $D^r \times D^{n-r}$ deformation retracts on $S^{r-1} \times D^{n-r} \cup D^r \times 0$. This gives a very close connection between handle decompositions and cell complexes. In particular, we deduce the following from Corollary 2.3.

Proposition 3.1. *If W is closed, it has the homotopy type of a finite CW complex. In general, $(W, \partial_- W)$ has the homotopy type of a finite CW pair.*

Proof. The first statement follows by taking a normalised handle decomposition of W and replacing each handle by an equivalent cell. In fact it would not be difficult to show (using the methods of Chapter 1) that in this case W is even homeomorphic to an appropriate finite CW complex.

For the second statement, note that by the first, we can regard $\partial_- W$ as a finite cell complex, and again apply Corollary 2.3. □

We now discuss duality. Observe that with f, $-f$ is also non-degenerate. Its critical points coincide with those of f, but if f has index λ at 0, it has locally the form

$$f(x) = c - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2$$

and $-f$ has index $n - \lambda$. Using the correspondence (Theorems 1.11 and 1.15) between non-degenerate functions and handle decompositions, we find the following.

Proposition 3.2. *Suppose W has a handle decomposition on $\partial_- W$ with α_r r-handles for $0 \leqq r \leqq n$. Then it also has one on $\partial_+ W$, with α_r $(n-r)$-handles*

If we ignore corners, we may identify the handles in the two cases, and observe that in the reversal. a- and b-spheres are interchanged. Now up to homotopy we may replace handles by cells. For homology, we have chain groups

$$C_r(W, \partial_- W) = \oplus \mathbb{Z} \quad (\alpha_r \text{ times});$$

we must calculate the boundary homomorphism

$$\partial \colon C_{r+1}(W, \partial_- W) \to C_r(W, \partial_- W).$$

This is determined by incidence numbers, one for each r- and $(r+1)$-handle.

Lemma 3.3. *The incidence number of handles h^{r+1} and h^r equals the intersection number of the a-sphere S^r of h^{r+1} and the b-sphere S^{n-r-1} of h^r.*

Proof. Here we shall write $W_{r+1/2} = (\partial_- W \times I) \cup$ all s-handles for $s \leqq r$, and $M = \partial_+ W_{r+1/2}$: the intersection number is taken in M, where we use Lemma 2.2 and add all r-handles simultaneously. A word about signs: the cells in the cell complex $(D^r \times 0)$ are arbitrarily oriented; this induces orientations of their bounding a-spheres S^{r-1} and of the *normal bundles* of their b -spheres. If an a-sphere S^r and a b-sphere S^{n-r-1} meet transversely at a point, we take the sign $+$ or $-$ according as the orientation of S^r does or does not agree with that in the normal bundle of S^{n-r-1}: thus orientability of W is irrelevant. If, though, W (and hence M) *is* oriented, orienting the normal bundle of a belt sphere is equivalent to orienting the sphere, and we can count multiplicities in the usual way.

Now we may suppose that S^r meets S^{n-r-1} transversely: then the intersection number agrees with the (local) degree of the projection of S^r on the normal disc D^r; but this degree is the required incidence number. ☐

Theorem 3.4. *(Duality Theorem)*
If W is orientable, $H_r(W, \partial_- W) \cong H^{n-r}(W, \partial_+ W)$.

Proof. By Proposition 3.2 we can identify the chain groups of $(W, \partial_- W)$ with the chain or cochain groups of $(W, \partial_+ W)$. By Lemma 3.3 the incidence numbers are the same up to sign (only a-spheres and b-spheres are interchanged) and the isomorphism identifies the one boundary with the other coboundary. ☐

Corollary 3.5 (Poincaré Duality)**.** *If $\partial W = \varnothing$, $H_r(W) \cong H^{n-r}(W)$*

Corollary 3.6. *(Lefschetz Duality)*

$$H_r(W) \cong H^{n-r}(W, \partial W)$$
$$H^r(W) \cong H_{n-r}(W, \partial W).$$

The proof above is surprisingly reminiscent of the earliest proofs of the result, but of course is only valid for compact smooth manifolds.

As a special case of homology groups, we mention connectivity. We retain the notation of Lemma 3.3. Observe that the a-sphere S^{-1} of a 0-handle is the empty set; in fact a 0-handle consists precisely of an n-disc, disjoint from $\partial_- W \times I$. Now the a-sphere S^0 of a 1-handle is a pair of points: these may or may not be in the same component of $W^{1/2}$. If not, the 1-handle connects the two components; but if they are, the corresponding handle does not affect connectivity.

If $\partial_- W$ is non-orientable then so, of course, is W. If, however, $\partial_- W$ is orientable, so is $W^{1/2}$, since adding a disjoint set of discs has no effect. Nor does adding a set of 1-handles which connect different components of $W^{1/2}$ (we are thinking of 1-handles as being added in turn, not simultaneously). However, the attaching map for a 1-handle is a map of $S^0 \times D^{n-1}$ - i.e., of a pair of discs. If these are mapped into the same component of $W^{1/2}$ with opposite orientations, then the orientation of $W^{1/2}$ can be extended over the handle; but if with the same orientation, $W^{1\frac{1}{2}}$ is non-orientable. Thus if, say, $W^{1/2}$ is connected and orientable, we may speak of orientable and of non-orientable 1-handles. It is now easy to see that r-handles for $r \neq 1$ do not affect orientability; for they introduce no new (potentially orientation-reversing) elements of the fundamental group.

This illustrates how the addition of handles affects W; we next discuss what happens to the boundary on addition of a handle.

Definition 3.7. Let M^{n-1} be a manifold, $f\colon S^{r-1} \times D^{n-r} \to M$ an imbedding. The operation of removing the interior of the image of f, and attaching $D^r \times S^{n-r-1}$ to the result by $f|S^{r-1} \times S^{n-r-1}$ is called a *spherical modification* of M, of type $(r, n-r)$.

We observe the following;

(S1) The effect of a spherical modification is determined by f - even by the diffeotopy class of f (by Theorem I, 4.4).

(S2) The modification gives a manifold M' with the same boundary as M: in particular, if M is closed so is M'.

(S3) Set $W = (M \times I \cup_f h^r)$. The manifold W (with corner, if M has a boundary) thus has M, M' as $\partial_- W, \partial_+ W$; we may call it the *supporting manifold* of the modification. Also, $\partial_c W = \partial M \times I$.

(S4) If M' is obtained from M by a spherical modification of type $(r, n-r)$, we can obtain M from M' by one of type $(n-r, r)$. This is essentially the remark that we made above in discussing duality. We have the same supporting manifold for both modifications.

We recall that if a cobordism W has $M = \partial_- W$, $N = \partial_+ W$, M and N are called *cobordant*.

If M^{n-1}, N^{n-1} are oriented, they are *cobordant in the oriented sense* if W^n is oriented, and W induces the given orientation of M, and the negative of the given one on N - this is usually written as $\partial W = M \cup (-N)$.

Proposition 3.8. *M^{n-1}, N^{n-1} are cobordant if and only if one may be obtained form the other by a series of spherical modifications; if oriented, they are cobordant in the oriented sense if and only if, in addition, the modifications of types $(1, n-1)$ and $(n-1, 1)$ all correspond to 1-handles of the orientable type.*

Proof. The first statement is an immediate consequence of (S3) and Complement 1.13; the second follows from that and the discussion of orientability above. □

Finally let M' be obtained from M by an $(r, n-r)$-modification: we wish to discuss homology and homotopy. There are two approaches: to use the supporting manifold $W = (M \times I) \cup_f h^r$ or the intersection $X = M \cap M'$. For, up to homotopy, W is obtained from M by attaching an r-cell, and form M' by attaching an $(n-r)$-cell. On the other and, M is obtained from X by attaching an $(n-r)$- and an $(n-1)$-cell, and M' from X by attaching an r-and an $(n-1)$-cell.

Proposition 3.9. *Let $r \leqq n-r$. Then M and M' have the same $(r-2)$-type (in particular, if $r \geqq 3$, the same fundamental group). If $r < n-r$, and x, ξ are the homology and homotopy classes of the a-sphere $f(S^{r-1} \times 0)$ in M, then*

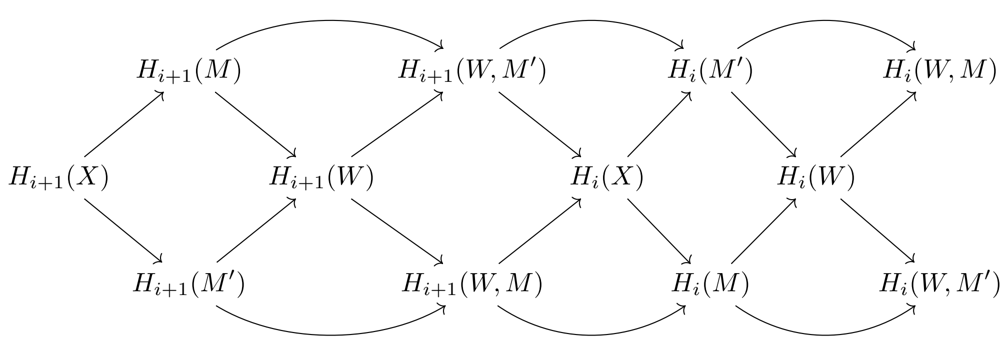
(i) $H_{r-1}(M')$ is the quotient of $H_{r-1}(M)$ by the subgroup generated by x.

(ii) If $r = 2$, $\pi_1(M')$ is the quotient of $\pi_1(M)$ by the normal subgroup generated by ξ.

(iii) If $r \geqq 3$, $\pi_r(M')$ is the quotient of $\pi_r(M)$ by the $\pi_1(M)$-submodule generated by ξ.

These all follow from standard properties of cell complexes. We can express the homology relations by a single diagram, as follows. Observe that the inclusions $(M', X) \subset (W, M \times I) \supset (W, M)$ induce isomorphisms of relative homology groups in dimensions $\neq n-1$. Indeed, excising most of X, they become $(D^r \times S^{n-r-1}, S^{r-1} \times S^{n-r-1}) \subset (D^r \times D^{n-r}, S^{r-1} \times D^{n-r})$ and both relative groups vanish except in dimensions r, $n-1$; in dimension r we have an isomorphism.

Proposition 3.10. *(Wall's "braid")*
We have the following exact sequences for $i < n-2$.



Proof. Identify $H_j(M', X) = H_j(W, M) \quad (j \leqq n-2)$ and $H_j(M, X) = H_j(W, M')$, dually. Then write out the exact homology sequences of the four pairs (M, X), (M', X), (W, M), and (W, M'). $\square$

Chapter 4

Modifying decompositions

In this chapter we discuss several modifications that can be made to handle decompositions: introduction or cancellation of a complementary pair of handles; addition of handles; replacement of a handle by one in a different dimension. These will be used below to obtain a minimal form of handle decomposition.

As the simplest case, we first discuss 0-handles. We may suppose that W is connected. Now if W has α_i i-handles, we know that $W^{1/2} \cong \partial_- W \times I \cup_{\alpha_0} D^n$. To this we add 1-handles, which must make it connected; moreover, a 1-handle affects connectivity only if its a-sphere S^0 has the two points in different components of $W^{1/2}$. Rearrange the 1-handles (Lemma 2.2) such that the first few each connect different components of $W^{1/2}$, till it is connected. Observe that for one of these, we have two manifolds with boundary, and a disc imbedded in the boundary of each. Attaching $D^{n-1} \times I$ is the same (I, 7) as glueing along the $(n-1)$-discs, i.e., forming the sum (Definition I, 7.9). Moreover, by (I, 7.11), for any manifold N^n, $N^n + D^n \cong N^n$. So the 0-handles are just cancelled out, and the various components of $\partial_- W \times I$ added together.

Proposition 4.1. *$W^{1/2}$ admits a handle presentation of the following kind.*

1. *If $\partial_- W = \varnothing$, there is one 0-handle D^n, and a number of 1-handles.*
2. *If $\partial_- W$ is connected, there are no 0-handles, but a number of 1-handles.*
3. *If $\partial_- W$ has components $M_{(i)}$, $1 \leqq i \leqq k$, there are no 0-handles, then $(k-1)$ 1-handles connecting the components to give $M_{(1)} \times I + \cdots + M_{(k)} \times I$, then a further number of 1-handles.*

Corollary 4.2. *The new presentation has,*

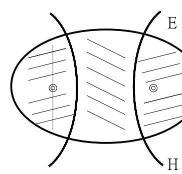
1. *if $\partial_- W = \varnothing$, one 0-handle and $(\alpha_1 - \alpha_0 + 1)$ 1-handles;*
2. *if $\partial_- W \neq \varnothing$, no 0-handle and $(\alpha_1 - \alpha_0)$ 1-handles.*

Proof. Each use of $N^n + D^n \cong N^n$ to simplify the decomposition removes just one 0-handle and one 1-handle. □

We next wish to discuss cancellation of handles. We first prove the simplest case.

Lemma 4.3. *Let* $\varphi\colon D^{n-r-1} \to D^{n-r}$ *be the imbedding, by stereographic projection from* $(0,\ldots,0,-1)$ *on the boundary of the upper hemisphere. Then* $S^r \times D^{n-r} \cup_{1\times\varphi} h^{r+1} \cong D^n$.

Proof. We first give the proof for $r = 0$, $n = 2$. Let E be the ellipse $\frac{1}{2}x^2 + y^2 = 1$ and H the confocal hyperbola $2x^2 - 2y^2 = 1$. Write Int and Ext for the (closed) interior and exterior. We shall show that $\mathrm{Int}E \cap \mathrm{Ext}H$ is obtained from $S^0 \times D^2$ by introducing a corner along $S^0 \times D^1$; that $\mathrm{Int}E \cap \mathrm{Ext}H$ is diffeomorphic to $D^1 \times D^1$, and that the attaching map $1 \times \varphi$ becomes the identity. It follows that the required manifold is diffeomorphic to $\mathrm{Int}E$, which is evidently diffeomorphic to D^2 (e.g., use the function $x^2 + y^2$ and apply Complement 1.9).



E meets H at $(\pm1, \pm1/\sqrt{2})$. Consider the component of $\mathrm{Int}E \cap \mathrm{Ext}H$ in $x > 0$; it has the focus $(1,0)$ as interior point. Rays through the focus define a vector field everywhere transverse to the boundary, which may therefore be used for straightening the corner. A smooth cross-section is given by $(x-1)^2 + y^2 = 1/4$, which meets the rays through the corner in $(1, \pm1/2)$. Thus the disc component is obtained from a disc by introducing corners at opposite ends of a diameter, as stated.

In $\mathrm{Int}E \cap \mathrm{Ext}H$ we use confocal coordinates. Each point (x,y) of the plane with $xy \neq 0$ lies on just two of

$$x^2/(\lambda+1) + y^2/\lambda = 1;$$

one hyperbola, given by $-1 < \lambda_1 < 0$, and one ellipse, given by $0 < \lambda_2$. However, these meet in 4 points. So we write $\mu^2 = a + \lambda_1$, $\nu^2 = \lambda_2$, and obtain

$$x = \mu\sqrt{1+\nu^2}$$
$$y = \nu\sqrt{1-\mu^2}$$

where the positive square roots are to be taken, and $-1 < \mu < 1$. It is easy to verify that this transformation is smooth, with nonzero Jacobian, 1 - 1 onto the whole plane except for $y = 0$, $x^2 \geqq 1$. Hence, in particular, it induces a diffeomorphism of the rectangle $|\mu| \leqq 1/\sqrt{2}$, $|\nu| \leqq 1$ onto $\mathrm{Int}E \cap \mathrm{Ext}H$, as required.

Now return to the case of general r and n, which is obtained by rotating the figures about x- and y-axes. Write

$$x = (x_1, \dots, x_{r+1}) \qquad y = (y_1, \dots, y_{n-r-1})$$
$$\mu = (\mu_1, \dots, \mu_{r+1}) \qquad \nu = (\nu_1, \dots, \nu_{n-r-1})$$

and $|x|^2 = \sum_1^{r+1} x_i^2$, etc. Then the transformation given by

$$x_i = \mu_i\sqrt{1+|\nu|^2}, \qquad y_i = \nu_i\sqrt{1-|\mu|^2}$$

induces a diffeomorphism of the $D^{r+1} \times D^{n-r-1}$ given by $|\mu|^2 \leqq 1/2$, $|\nu|^2 \leqq 1$ onto the intersection $\frac{1}{2}|x|^2 + |y|^2 \leqq 1$, $2|x|^2 - 2|y|^2 \leqq 1$.

Likewise in the intersection $\frac{1}{2}|x|^2 + |y|^2 \leqq 1$, $2|x|^2 - 2|y|^2 \leqq 1$, consider the field formed by rays through the r-sphere $y = 0$, $|x| = 1$ and perpendicular to it (and not produced beyond their intersection with $x = 0$). This certainly is a vector field (except on the sphere and on $x = 0$), and we easily see hat it is transverse to the boundary, so can be used for rounding the corner. Rounding it, we obtain the manifold $(|x| - 1)^2 + |y|^2 \leqq 1/4$, where the corner is to be introduced along $|x| = 1$, $|y| = 1/2$ (in fact $S^r \times S^{n-r-2}$).

Consider $S^r \times D^{n-r} \subset \mathbb{R}^{r+1} \times \mathbb{R}^{n-r-1} \times \mathbb{R}^1$ with coordinates (u, w, t), so $|u| = 1$, $|w|^2 + |t|^2 \leqq 1$. We define inverse diffeomorphism between this and the manifold above by

$$u = x/|x| \qquad w = 2y \qquad t = 2(|x| - 1)$$
$$x = u(1 + t/2) \qquad y = w/2.$$

Note that $|x|$ is nowhere zero, so it and its inverse are smooth. We also note that the corner $|x| = 1$, $|y| = 1/2$ becomes the locus $|w| = 1$, $t = 0$.

Finally we must identify the attaching map. The sphere $S^r \times 0$ given by $|\mu|^2 = 1/2$, $\nu = 0$ maps (via $x_i = \mu_i$) to $|x|^2 = 1/2$, $y = 0$, then rounding the corner multiplies x_i by $2^{-1/2}$ and leaves y at 0. Finally we obtain $u = x/|x| = \mu/|\mu|$ and $v = (w, t) = (0, -1)$; modulo the obvious identifications, in fact, we have the identity map. The attaching map is a tubular neighbourhood of this, and we note that a normal direction $\partial/\partial\nu_i$ maps to some positive multiple of $\partial/\partial v_i$; using the tubular neighbourhood theorem, it follows that the attaching map is, up to a diffeotopy, as stated. □

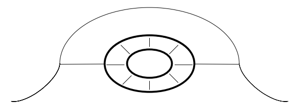
Note that if diffeomorphism is replaced by homeomorphism, this (and the next lemma) become much easier to prove; it was the necessity of rounding the corner systematically which led us to the formulae above.

Lemma 4.4. *Suppose that for $D^n \cup_f h^r \cup_g h^{r+1}$, the a-sphere of h^{r+1} meets the b-sphere of h^r transversely in one point. Then*

(i) $D^n \cup_f h^r \cong S^r \times D^{n-r}$

(ii) The diffeomorphism f can be so chosen that g becomes the map $1 \times \varphi$ of Lemma 4.3, and so $D^n \cup_f h^r \cup_g h^{r+1} \cong D^n$.

Proof. We first normalise (Theorem 2.5) so that we can write $g_-(D^r_- \times D^{n-r-1}) \subset \partial D^n$. Now $g_+ : D^r_+ \times D^{n-r-1} \to D^r \times S^{n-r-1}$ is of the form $\varphi_r^{-1} \times \varphi_{n-r-1}$ (in normal position, g is a product map; it is isotopic to the particular form shown, by the Disc Theorem). Also by the disc theorem, $g_-(D^r_- \times D^{n-r-1})$ is isotopic



to any other imbedding with the same orientation (for this manifold with corner is contained in a slightly larger disc, which can be constructed using a tubular neighbourhood of the corner and we use the uniqueness of that disc). This determines $g_+(S^{r-1} \times D^{n-r-1})$, hence also $f(\partial D^r \times \varphi_{n-r-1}(D^{n-r-1}))$, which may thus be put in a standard position. Applying the tubular neighbourhood theorem to this, we see that f, too, is essentially unique. Thus to prove (i), the existence of an f with the required property will suffice: we introduce a corner on D^n to make it $D^r \times D^{n-r}$, and take for f the inclusion of $\partial D^r \times D^{n-r}$. The result is $S^r \times D^{n-r}$, and g can be taken as $1 \times \varphi$. Now (ii) follows, also since the pair (f, g) was essentially unique. □

Theorem 4.5. *(The Cancellation Theorem) Suppose that for $M^n \cup_f h^r \cup_g h^{r+1}$, the a-sphere of h^{r+1} meets the b-sphere of h^r transversely in one point. Then we can suppose $\partial_+ M$ contains a disc D^{n-1} to which both handles are added. Thus we can write $M^n \cong N^n + D^n$, with the handles added to D^n, and so $M^n \cup h^r \cup h^{r+1} \cong N^n + (D^n \cup h^r \cup h^{r+1}) \cong N^n + D^n \cong M^n$.*

Proof. First normalise as for Lemma 4.4. Then the image of g_- is contained in a disc (as before); and similarly the image of f is contained in a tubular neighbourhood of the boundary of this disc, which merely extends it to a larger disc. The rest follows from the lemma. □

Definition 4.6. A pair of handle of consecutive dimensions, with the a-sphere of the second meeting the b-sphere of the first transversely in one point, is called a *complementary pair.*

We can thus paraphrase Theorem 4.5 briefly by saying that a complementary pair of handles may always be cancelled. The converse result is now trivial.

Theorem 4.7. *At any point of a handle decomposition of a manifold, a complementary pair of handles can be introduced.*

Proof. "At any point" means when we have constructed some manifold M, say. Now $M \cong M + D$ by Proposition I, 7.10 and by Lemma 4.3, we can add a complementary pair of handles to D, hence also to M.

We observe that adding two complementary handles in succession to M has the effect on $V = \partial_+ M$ of performing consecutively spherical modifications of types $(r, n-r)$ - leading to W, say -and $(r+1, n-r-1)$ - returning to V. 'Reversing' the second of these shows that we can also go from V to W by a modification of type $(n-r-1, r+1)$. The condition on the first modification necessary for this replacement to be possible was the existence of a complementary handle; arguing as in Lemma 4.4, we see that this is equivalent to requiring the a-sphere to span an r-disc in V, such that the inward normal vector to the sphere in the disc agrees with the first vector of the chosen normal framing of the a-sphere. □

We now discuss "addition" of handles - this is to be understood in a homotopy sense. Since $\partial_+ M$ need not be simply-connected, an $(r-1)$-sphere in it does not necessarily have a well-defined homotopy class. This ambiguity may be resolved by introducing as further structure a base-point * in $\partial_+ M$, and for each handle with attaching map $f\colon \partial D^r \times D^{m-r} \to \partial_+ M$ a path in $\partial_+ M$ from * to $f(1 \times 0)$: the homotopy class of $\bar{f}$ may be defined in an obvious way. Of course we shall look for results which do not depend much on the choice of path.

Theorem 4.8. *(Handle Addition Theorem) Suppose $\partial_+ W = M$ connected, $2 \leqq r \leqq m-2$. Let $f, g\colon \partial D^r \times D^{m-r} \to M$ be disjoint imbeddings, determining homotopy classes $\alpha, \beta \in \pi_{r-1}(M)$; let $\gamma \in \pi_1(M)$. Then there are imbeddings $h_+, h_-\colon \partial D^r \times D^{m-r} \to M$, disjoint from f, and determining $\beta + \alpha^\gamma, \beta - \alpha^\gamma \in \pi_{r-1}(M)$, such that $W \cup_f h^r \cup_g h^r \cong W \cup_f h^r \cup_{h_\varepsilon} h^r$ (for $\varepsilon = \pm$).*

Proof. We observe that α injects to zero in $W \cup_f h^r$; the idea of the proof is to deform the second handle 'across' the first, by a diffeotopy of the attaching map in $\partial_+(M \cup_f h^r)$; we know that this will not affect the diffeomorphism of the result.

We have supposed M connected, so there is a path λ joining $f(1 \times 1)$ and $g(1 \times 1)$. Notice that this path may be taken in any homotopy class. By the general position arguments (Part II Chapter 4), we can make the path an imbedding, disjoint form the images of $\bar{f}$ and $\bar{g}$; we can choose it to start along the outward normals to Imf and Img, and finally we can deform it off tubular neighbourhoods of Im$\bar{f}$ and Im$\bar{g}$, so that it meets Imf and Img only at its ends. Choose two normal framings $e_1, \dots, e_{m-2}$ for λ so that $e_1, \dots, e_{r-1}$ give the standard orientation of $g(S^{r-1} \times 1)$ at $g(1 \times 1)$ and both possible orientations of $f(S^{r-1} \times 1)$ at $f(1 \times 1)$: this is possible since $r \leqq m-2$: and choose a Riemannian metric in which $f(S^{r-1} \times 1)$ and $g(S^{r-1} \times 1)$ are totally geodesic (see I, 3.15). Then exponentiating normal vectors to λ gives an imbedding $\varphi'\colon I \times D^{r-1} \to M$ with $\varphi'(0 \times D^{r-1}) \subset g(S^{r-1} \times 1)$, $\varphi'(1 \times D^{r-1}) \subset f(S^{r-1} \times 1)$. Extend λ by a diameter of $D^{r-1} \times 1$ in $\partial_+(M \cup_f h^r)$, and φ' correspondingly to an imbedding

$\varphi\colon [0,2] \times D^{r-1} \to \partial_+(M \cup_f h^r)$. We now define an isotopy of $\bar{g}$ by

$$\left.\begin{aligned} \bar{g}_t(x) = x \quad \text{if} \quad x \notin \varphi(0 \times D^{r-1}) \\ \bar{g}_t\varphi(0,y) = \varphi(2tB_P(1-|y|), y) \end{aligned}\right\}$$

the properties of the bump function ensures that these fit to give a smooth diffeotopy. This 'pulls' the cell $\varphi(0 \times D^{r-1}) \subset g(S^{r-1} \times 1)$ across part of the disc $D^r \times 1$, covering the central point. Since $g(S^{r-1} \times 0)$ is diffeotopic to $g(S^{r-1} \times 1)$, we also obtain a diffeotopy of $\bar{g}$, which we can extend to one of g such that the final imbedding h is disjoint from $0 \times S^{n-r-1}$. But we can think of the (f-) handle as shrunk to a small neighbourhood of this b-sphere (c.f. proof of 2.5), so $h(S^{r-1} \times D^{n-r})$ lies in$M$ again.

Since our diffeotopy has (clearly) degree 1 on the attached cell, the homology class of h is that of g plus or minus that of f, the sign depending on an orientation chosen earlier. The same applies to homotopy, except for consideration of base points. But freedom of choice of homotopy class of λ is equivalent to the freedom of choice of γ in the Theorem. □

Remark 4.9. We could also discuss the homotopy classes in $\pi_r(W \cup_f h^r \cup_g h^r, W)$ represented by handles; these also are added, in the same way.

Chapter 5

Simplifying decompositions

In the last chapter we gave a method of simplifying handle decompositions under geometric assumptions. We shall now obtain some corresponding results under algebraic hypothesis: this will enable us to operate with handles using only homotopy data. There are several ways of applying the Cancellation Theorem 4.5: we start with the most direct. It is interesting to note that this closely resembles a technique of Whitehead, with CW complexes.

Theorem 5.1. *Suppose* $n \geqq 2r+3$, $W^n = M \times I \cup h^r \cup lh^{r+1}$, *and* $\pi_r(W, M) = 0$. *Then* $W \cong M \times I \cup lh^{r+1} \cup h^{r+2}$.

Proof. The case $r = 0$ follows from Proposition 4.1; otherwise we may suppose M connected.

We identify h^r with $D^r \times D^{m-r}$. Since $n \geqq 2r+2$, we can perform a diffeotopy to ensure that the attaching maps of the h^{r+1} avoid $D^r \times 1$. The disc $D^r \times 1$ determines an element of $\pi_r(W, M)$, which is zero by the hypothesis. Hence this disc is homotopic in W (relative to its boundary) to one in M; i.e., there is a map $F\colon D^{r+1} \to W$, which takes the upper hemisphere of S^r onto $D^r \times 1$ and the lower into M. Since $n \geqq 2r+3$, we may suppose that $\mathrm{Im}F$ is disjoint from the cores of the handles (of dimensions r and $(r+1)$). We can therefore also deform it off tubular neighbourhoods of the cores, so that eventually $\mathrm{Im}F \subset \partial_+ W$. We may suppose $F|S^r$ an imbedding of S^r in $\partial_+ W$: this imbedding is homotopic to zero, hence also diffeotopic (we again use $n \geqq 2r+2$). So by Theorem 4.7, we can use it for the a-sphere of the first of a complementary pair of handles h_A^{r+1}, h_B^{r+2}, where h_A^{r+1} is disjoint from the other h^{r+1}. But h_A^{r+1} is also complementary to h^r, so

$$\begin{aligned} W &\cong M \times I \cup h^r \cup lh^{r+1} \cup (h_A^{r+1} \cup h_B^{r+2}) && \text{(Theorem 4.7)} \\ &\cong M \times I \cup (h^r \cup h_A^{r+1}) \cup lh^{r+1} \cup h_B^{r+2} \\ &\cong M \times I \cup lh^{r+1} \cup h_B^{r+2} && \text{(Theorem 4.5).} \end{aligned}$$

□

Corollary 5.2. *We can replace $M \times I$ in the theorem by V, provided $\partial_+ V \subset V$ induces $\pi_1(\partial_+ V) \cong \pi 1(V)$.*

Proof. Set $M_1 = \partial_+ V$; we may suppose $W = V \cup (M \times I) \cup$ handles; write W_1 for the closure of $W \setminus V$. Then it is enough to show that $\pi_r(W_1, M_1) = 0$. Now if $r = 1$, by van Kampen's theorem,

$$\pi_1(W) \cong \pi_1(V) *_{\pi_1(M_1)} \pi_1(W_1) \cong \pi_1(W_1),$$

and if $r \geqq 2$, by the Hurewicz theorem,

$$\pi_r(W_1, M_1) \cong H_r(\tilde{W}_1, \tilde{M}_1), \qquad \pi_r(W, V) \cong H_r(\tilde{W}, \tilde{V}),$$

(the universal covers of W_1, V, M_1 are induced from that of W, since under our assumption, the fundamental groups map isomorphically), and we can now use excision. □

Corollary 5.3. *If $W = V \cup kh^r \cup lh^{r+1}$, $\pi_r(W, V) = 0$, $\pi_r(\partial_+ V) \cong \pi_1(V)$, $n \geqq 2r + 3$, then $W \cong V \cup lh^{r+1} \cup kh^{r+2}$.*

Proof. Write $V_1 = V \cup (k-1)h^r$. Since $\pi_r(W, V)$ and $\pi_{r-1}(V_1, V)$ vanish, so does $\pi_r(W, V_1)$. Also, if $V = M \times I$, $M_1 = \partial_+ V_1$, then $V_1 \cong M_1 \times I \cup (k-1)h^{n-r}$, so $\pi_1(M_1) \cong \pi_1(V_1)$ if $n \geqq r + 3$. For a general V, we use van Kampen's theorem, as above, to deduce $\pi_1(\partial_+ V_1) \cong \pi_1(V_1)$. Now by Corollary 5.2, we have $W \cong V_1 \cup lh^{r+1} \cup h^{r+2}$ so the result follows by induction on k. □

Corollary 5.4. *Suppose (W, V) r-connected, $\pi_1(\partial_+ V_1) \cong \pi_1(V_1)$, $n \geqq 2r + 3$. Then W has a handle decomposition on V with no i-handles for $i \leqq r$.*

Proof. Use Corollary 5.3 repeatedly to replace i-handles by $(i+2)$-handles for $i = 0, \ldots, r$. □

Remark 5.5. We can tighten up the proof of Theorem 5.1 to cover also the case $n = 2r + 2, r \neq 1$. (In fact the only point to be watched is the deformation of F off the cores of the h^{r+1}.) But we obtain a more general result below, by a different method, for $r \geqq 2$. To introduce this method, we first consider a simple special case (the first, historically, too - due to S. Smale): we observe that we succeed in cancelling handles, not merely replacing some by others.

Theorem 5.6. *Suppose $\partial_+ V^n$ simply connected, $r \geqq 3$, $(n - r) \geqq 4$. Let $W = V \cup h^r \cup kh^{r+1}$ and $H_r(W, V) = 0$. Then $W \cong V \cup (k-1)h^{r+1}$.*

Proof. Let $n_1, \ldots, n_k$ be the intersection numbers of the a-spheres of the $(r+1)$-handles with the b-sphere of h^r. By Theorem 4.8, we can 'add' and 'subtract' the handles; hence if n_i, n_j are nonzero, say $n_i > n_j > 0$, we can replace n_i by $n_i - n_j$. Hence by induction on $\sum_1^k |n_i|$ - we may suppose all the n_i zero except one - say n_1. By Lemma 3.3, the assumption $H_r(W, V) = 0$ now implies $n_1 = \pm 1$.

Now use Theorem 2.5 to normalise the handle decomposition. Then the a-spheres of h_1^{r+1} and the b-spheres of h^r, both of dimension at least 3, meet transversely and have intersection number ± 1 in $\partial_+(V \cup h^r)$, which by Proposition 3.9 is simply connected since $\partial_+ V$ is. Hence by Part III we can perform a diffeotopy to reduce the number of intersections to one. But then h^r and h^{r+1} are complementary, so can be cancelled by Theorem 4.5. □

We now consider the general case and, in particular, abandon simple-connectivity. This is more technical, and we shall eventually refer to the notion of simple homotopy theory. We first state the general conditions which we need to assume.

Hypothesis 5.7. $W^n = V^n \cup kh^r \cup lh^{r+1}$, $\pi_r(W, V) = 0$, $\pi_1(\partial_+ V) \cong \pi_1(V)$. We have $r \geqq 2$, $n - r \geqq 4$; or $r \geqq 3$, $n - r = 3$, and $\pi_1(\partial_+ W) \cong \pi_1(W)$. Set $M = \partial_+(V \cup kh^r)$, $\pi = \pi_1(M)$ and $\Lambda = \mathbb{Z}[\pi]$.

The ring Λ consists of finite (formal) linear combinations, with integer coefficients, of elements of π, with the obvious multiplication. Using Proposition 3.9, $\pi_1(V) \cong \pi_1(M) \cong \pi_1(\partial_+ W) \cong \pi_1(V \cup kh^r) \cong \pi_1(W)$, (note if $r = 2$ that our hypothesis implies $\pi_1(V) \cong \pi_1(W)$) and the isomorphisms are induced by inclusion maps. We use tilde for all universal covering spaces. By the Hurewicz theorem, $\pi_r(W, V) \cong H_r(\tilde{W}, \tilde{V})$, so our hypothesis gives some information about the chain map in the universal covering space. To use this, we need the lemma below; first, however, we need some notation.

Let * be a base point in $V \cap \partial_+ W$ (hence in M): join by paths in M to the a-spheres of the h^{r+1} and the b-spheres of the h^r: all these lie in M. Now to each intersection P of the b-sphere of an r-handle h_j^r with the a-sphere of h_i^{r+1} we assign the element $g_P \in \pi$ represented by the path from * to the a-sphere, round this to P, along the b-sphere, and back by the chosen path of the b-sphere. We also set $\varepsilon_P = \pm 1$ according as the orientations agree or differ (c.f. Lemma 3.3).

Label the handles h_j $(1 \leqq j \leqq k)$, h_i^{r+1} $(1 \leqq i \leqq l)$; let $\tilde{h}_j^r$, $\tilde{h}_i^{r+1}$ be the handles above them in $\tilde{W}$, determined by some lifting of *, and lifting the chosen paths, and let $\tilde{e}_j^r$, $\tilde{e}_i^{r+1}$ be the corresponding chains of $(\tilde{W}, \tilde{V})$. We note that all the handles of (W, V) are of the form $g\tilde{h}_j^r$, $g\tilde{h}_i^{r+1}$ for various $g \in \pi$; these are all distinct, since we took the universal cover; so that $C_r(\tilde{W}, \tilde{V})$ and $C_{r+1}(\tilde{W}, \tilde{V})$ are the free Λ-modules with bases the $\{\tilde{e}_j^r\}$, $\{\tilde{e}_i^{r+1}\}$.

Lemma 5.8. *(i) We have $\partial \tilde{e}_i^{r+1} = \sum_{P,j} \varepsilon_P g_P \tilde{e}_j^r$, where the sum is over intersections P of the a-sphere of h_i^{r+1} with the b-sphere of h_j^r.*

(ii) If the coefficient of $\tilde{e}_j^r$ in $\tilde{e}_i^{r+1}$ is ± 1, we can perform a diffeotopy to make h_i^{r+1} complementary to h_j^r.

Proof. (i) If $\tilde{P}$ is the point of $\tilde{h}_i^{r+1}$ lying over P, it represents the intersection of the a-sphere of $\tilde{h}_i^{r+1}$ with the b-sphere of some $g'\tilde{h}_j^r$. If we lift he defining path of g_P, we see that $g' = g_P$. The result now follows from Lemma 3.3 (which did not use compactness).
(ii) It follows from the assumption that, with one exception, we can collect intersections of the two spheres into pairs (P, Q) with $g_P = g_Q$, $\varepsilon_P = -\varepsilon_Q$. The

result will follow if we show how to remove the intersections P and Q. Observe that the spheres - say S_a^r and S_b^{n-r-1} - have complementary dimension in M, and each has dimension $\geqq 2$. If we join P to Q by an arc in S_a and one in S_b we obtain a circle; moreover since $g_P = g_Q$, this circle is null-homotopic in M (which is of dimension $\geqq 5$) and so it bounds a disc. If we can make this disc disjoint from S_a and S_b, the usual method of removing intersections (due to Whitney; see III) applies, and we can remove P and Q; this can certainly be achieved if the codimensions r and $n-r-1$ are $\geqq 3$.

Now consider the case $r = 2$, $n - r > 3$. Here the disc may be supposed disjoint from S_a; also we note that the process of constructing the disc gives first (when $\varepsilon_P = -\varepsilon_Q$) an annulus which pushes the circle off $S_a \cup S_b$. So the result will follow if we show $\pi_1(M \setminus S_b) \cong \pi_1(M)$. The proof of this is sufficiently illustrated by the case $k = 1$; here we may identify $M \setminus S_b$ with $\partial_+ V \setminus S^1$, where S^1 is the a-sphere of h^2. So $\pi_1(M \setminus S_b) \cong \pi_1(\partial_+ V \setminus S^1) \cong \pi_1(\partial_+ V) \cong \pi_1(M)$ (for the codimension of S^1 is $\geqq 4$). If $r = n - 3$, $r > 2$, there is a similar argument using the hypothesis $\pi_1(\partial_+ W) \cong \pi_1(W)$. The proof breaks down completely if $r = 2$, $n = 5$. □

Remark 5.9. The same argument enables us to extend Theorem 5.6 to the case $r = 2$, $r = n - 3$.

Theorem 5.10. *Theorem 5.1 (and its corollaries) hold whenever $n \geqq r + 4$; also if $n = r + 3$ provided $r \neq 1, 2$ and $\pi_1(\partial_+ W) \cong \pi_1(W)$.*

Proof. Since $H_r(\tilde{W}, \tilde{V}) = 0$, $\partial\colon C_{r+1}(\tilde{W}, \tilde{V}) \to C_r(\tilde{W}, \tilde{V})$ is onto, so we can solve $\partial(\sum \lambda_i \tilde{e}_i^{r+1}) = \tilde{e}^r$. By Theorem 4.7 we can introduce a complementary pair of handles h_A^{r+1}, h_B^{r+2}; by applying Theorem 4.8 repeatedly, we can 'add' to the a-sphere of h_A^{r+1} any Λ-linear combination of the a-sphere of the other h^{r+1}. So we may suppose $\partial(\tilde{e}_A^{r+1}) = \tilde{e}^r$. Now by Lemma 5.8 (ii), we can perform a diffeotopy to make h_A^{r+1} complementary to h^r, and by Theorem 4.5 we can then cancel these two handles. □

Theorem 5.11. *Assume Hypothesis 5.7, and that the inclusion of V in W is a simple homotopy equivalence (so $k = l$). Then $W \cong V$.*

Proof. We shall not discuss here the definition of simple homotopy type, not the equivalence of definitions via triangulations and handle decompositions, but instead assume that our hypothesis is equivalent to assuming $\partial\colon C_{r+1}(\tilde{W}, \tilde{V}) \to C_r(\tilde{W}, \tilde{V})$ a simple isomorphism (that it is an isomorphism follows if the inclusion is any homotopy equivalence). Hence the matrix of ∂ can be reduced to a unit matrix by a sequence of moves of the following kinds:

(i) Add some multiple of a row to another row.

(ii) Multiply some row by an element of π, or by -1.

(iii) Take the direct sum of the matrix with (1).

But each of these can be induced by a change of the handle decomposition: (i) by handle addition (Theorem 4.8), (ii) by changing the path from * to an a-sphere, or the orientation of a cell, and (iii) by inserting a complementary pair of handles (Theorem 4.7). Thus we may assume that the matrix of ∂ is the identity, and $\partial \tilde{e}_i^{r+1} = \tilde{e}_i$. Now by Lemma 5.8 (ii), we can perform a diffeotopy on the a-sphere of h_i^{r+1} (leaving other handles fixed) to make h_i^{r+1} complementary to h_i^r. But then the handles are complementary in pairs, and can all be cancelled, by Theorem 4.5.

We observe that the proof of Theorem 5.6 shows that, in the simply-connected case, any matrix of determinant ± 1 can be reduced to the identity by moves (i) and (ii), so that if $\pi = \{1\}$, a homotopy equivalence is a simple homotopy equivalence. The same is also known to hold if $\pi \cong \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$, or if π is free or free abelian, or an elementary 2-group. $\square$

Chapter 6

The h-cobordism theorem

Definition 6.1. Let W be a cobordism. If the inclusions of $\partial_- W$, $\partial_+ W$ in W are homotopy equivalences, W is called an *h-cobordism*; if they are simple homotopy equivalences, W is called an *s-cobordism*

Theorem 6.2. *Assume that the inclusions of $\partial_- W$ in W is a homotopy equivalence and that the inclusion of $\partial_+ W$ induces an isomorphism of fundamental groups. Then*

(i) The inclusion of $\partial_+ W$ is a homotopy equivalence.

(ii) If either inclusion is a simple homotopy equivalence, so is the other, and if $n \geqq 6$, W is diffeomorphic to $\partial_- W \times I$.

Proof. By Corollary 5.3, W has a handle decomposition on $\partial_+ W$ with no 0- or 1-handles ($n \geqq 5$). Take the dual decomposition and apply Lemma 5.8: this says we may cancel the r-handles for $n \leqq n-4$, and leaves only those with $r = n-3$, $n-2$. Also, by elementary homology theory, there must be the same number k of handles of these two dimensions, and the chain complex of the universal cover consists of a single isomorphism $\partial\colon C_{n-2} \to C_{n-3}$. So W has a handle decomposition on $\partial_+ W$ with only 2- and 3-handles (the 2-handles attached trivially), and for this the matrix of $\partial_*\colon C_3 \to C_2$ is transpose conjugate (via $g \mapsto g^{-1}, g \in \pi$) of that of ∂, so ∂_* is also an isomorphism. Thus all $H_i(\tilde{W}, \widetilde{\partial_+ W}) = 0$, all $\pi_i(W, \partial_+ W) = 0$ by the Hurewicz theorem and (i) follows by the Whitehead theorem. [For $n \leqq 4$ we can use a more direct statement which is always valid.]

Now let the inclusion of $\partial_- W$ be a simple equivalence. Then ∂ is a simple isomorphism. If $n \geqq 6$, by Theorem 5.11, all handles may be cancelled, so $W \cong \partial_- W \times I$, and the result is proved. □

Corollary 6.3. *Suppose $W^n (n \geqq 6)$ a simply connected h-cobordism. Then $W \cong \partial_- W \times I$.*

The same argument will give us somewhat more general results if we relax the compactness condition. For example, let V be a submanifold of W such that there is a diffeomorphism $\varphi\colon V \cong \partial_- V \times I$. Then as in Lemma 1.2, we can find a non-degenerate function on W whose restriction to V has no critical points; the proof of Lemma 1.2 is only changed by using the given product structure to define g near V. We can now carry out all the handle decomposition and cancellation arguments in $W \setminus V$. Write N for a tubular neighbourhood of V in W, $\mathring{N}$ for its interior, $X = W \setminus \mathring{N}$ and $Y = N \cap X = \partial_c N = \partial_c X$.

Theorem 6.4. *With the above notations, suppose X an s-cobordism. Then φ can be extended to a diffeomorphism of W on $\partial_- W \times I$.*

Proof. As in Theorem 6.2, we can cancel all the handles in X. □

Lemma 6.5. *With the above notations, suppose W^n, V^{n-c} h-cobordisms, and $c \geqq 3$. Then X is an h-cobordism.*

Proof. Since $c \geqq 3$ is the codimension of V in W (and of $\partial_- V$ in $\partial_- W$, $\partial_+ V$ in $\partial_+ W$), removing V does not alter the fundamental group. So it is enough to check that $\partial_- X \subset X$ induces isomorphisms of homology in the universal cover, by Whitehead's theorem. This reduces the problem to the case when W is simply connected.

Now since $\partial_- V$ is a deformation retract of V, and N is a disc bundle, $\partial_- N$ is a deformation retract of N, also of $\partial_- N \cup Y$. Hence $0 = H_*(N, \partial_- N \cup Y) \cong H_*(W, X \cup \partial_- W)$ by excision.

But $0 = H_*(W, \partial_- W)$, so using the homology exact sequence of the triple $\partial_- W \subset X \cup \partial_- W \subset W$, we deduce $0 = H_*(X \cup \partial_- W, \partial_- W) \cong H_*(X, \partial_- X)$ by excision. The result follows. □

Corollary 6.6. *Suppose W^n a simply-connected h-cobordism, $n \geqq 6$, V^{n-c} a submanifold, $c \geqq 3$, such that $V^{n-c} \cong \partial_- V \times I$. Then $(W, V) \cong (\partial_- W, \partial_- V) \times I$.*

Proof. Since $W \setminus V$ is simply-connected, the lemma shows that 6.4 applies. □

Corollary 6.7. *Two h-cobordant pairs of homotopy spheres $(T_i^{n+c}, T_i^n)(i = 0, 1)$ with $n \geqq 5$, $c \geqq 3$ are diffeomorphic.*

Proof. By Corollary 6.3, the h-cobordism of the T_i^n is a product, so the result follows from Corollary 6.6. □

We also have a slight generalisation of 6.2.

Theorem 6.8. *Suppose $V^n \subset W^n$ a simple homotopy equivalence, that $n \geqq 6$, $\partial_- V = \partial_- W$, and that $\partial_+ V \subset V$, $\partial_+ W \subset W$ induce isomorphisms of fundamental groups. Then $W = V \cup \partial_+ V \times I$.*

Proof. Let $M = \partial_+ V$, X be the closure of $W \setminus V$. Then

$$\pi_1(W) \cong \pi_1(X)_{*\pi_1(M)} \pi_1(V) \cong \pi_1(X).$$

By Corollary 5.3, X has a handle decomposition on $\partial_+ W$ with no 0- or 1-handles. So W has one on V with no $(n-1)$- or n-handles. Applying Theorem 5.11 repeatedly, we can get rid of i-handles for $i < n-3$. The result now follows from Theorem 5.11. □

Corollary 6.9 (Disc Bundle Theorem)**.** *Suppose M^{n-c} a submanifold of W^n, $\partial M = \varnothing$, $c \geqq 3$, $n \geqq 6$, $M \subset W$ a simple homotopy equivalence, and $\pi_1(\partial_+ W) \cong \pi_1(W)$. Then W has the structure of a disc bundle with M as zero cross-section.*

Proof. Let V be a tubular neighbourhood of M, then 6.8 applies. $\pi_1(\partial_+ V) \cong \pi_1(V)$ since $c \geqq 3$ (it can also happen for $c = 1, 2$). □

Corollary 6.10. *If W^n is contractible, $n \geqq 6$, $\pi_1(\partial W) = 0$, then $W^n \cong D^n$.*

Proof. Take M a point in 6.9. □

Corollary 6.11 (Poincaré Conjecture)**.** *If T^n is a homotopy sphere, $n \geqq 6$, then T^n may be obtained by glueing two discs together along the boundary. Thus T^n is homeomorphic to S^n.*

Proof. Let W^n be the closure of the complement of a disc D^n in T^n. Then W is homotopic to $T^n \setminus \{point\}$, so is simply-connected, and its reduced homology groups vanish, so W is contractible. By 6.10, $W^n \cong D^n$. The last remark follows since any homeomorphism of S^{n-1} can be extended (taking the cone) to D^n. □

Remark 6.12. The result follows from 4.1 if $n \leqq 2$, and holds if $n = 5$, when we shall show later that any T^5 bounds a contractible W^6. The cases $n = 3$, $n = 4$ are unsolved.[1]

Corollary 6.13. *Suppose M_i^m compact, $\partial M_i^m = \varnothing (i = 1, 2)$, $f\colon M_1 \to M_2$ a simple homotopy equivalence and $2c \geqq m$. Then $M_2 \times D^c$ is a disc bundle over M_1.*

Proof. If $c < 3$, $m \leqq 1$, $M_1 \cong M_2 \cong S^1$ (or $\{point\}$) and the result is trivial. Now let $c \geqq 3$. Then by Haefliger's theorem (III), we can approximate f by an imbedding of M_1 in $M_2 \times D^c$. The result now follows from the Theorem. □

Corollary 6.14. *Suppose in addition that $c \geqq m+1$ and $f^*(T_{M_2}+1) \cong T_{M_1}+1$. Then $M_1 \times D^c \cong M_2 \times D^c$.*

Proof. Under these conditions the normal bundle of $g(M_1)$ in $M_2 \times D^c$ is stably trivial and stable, hence trivial. □

Theorem 6.15. *Let T^{n-c} be a homotopy sphere in $S^n (n \geqq 6, c \geqq 3)$, N a tubular neighbourhood, V the closure of its complement. Then V is diffeomorphic to $S^{c-1} \times D^{n-c+1}$.*

[1] The topological case for $n = 4$ was proved by Michael Freedman in 1982. The case for $n = 3$ was proved by Grigori Perelman in 2003.

Proof. Let N' be a larger concentric tube, D^c a fibre, S^{c-1} its boundary. Since S^{c-1} bounds the contractible D^c, its normal bundle is trivial. We assert that the inclusion of S^{c-1} in V is a homotopy equivalence; indeed, both are simply-connected (V since S^n is and $c \geqq 3$, so $T \setminus S$ is also) and the complement of $V \cup D^c$ is the interior of $N \setminus D^c$, a cell bundle over a cell and so contractible. By duality, $V \cup D^c$ is contractible, and $0 = H_r(V \cup D^c, D^c) = H_r(V, V \cap D^c)$. But $V \cap D^c$ is an annulus with S^{c-1} as deformation retract, hence $H_r(V, S^{c-1}) = 0$.

If $c \neq n-1$, $\partial V = \partial N$ is simply-connected, and $n - c + 1 \geqq 3$, so the result follows from Theorem 6.8. If $c = n-1$, T is a circle, and unknots, so the result is trivial. □

Theorem 6.16. *Suppose W^n ($n \geqq 6$) such that $\partial_- W$, $\partial_+ W$ and W are simply-connected. Let $H_i(W, \partial_- W) \cong F + T$, where F is a free abelian group of rank β_i and T is a finite group with $\tau_{i+1/2}$ generators. Then W has a handle decomposition on $\partial_- W$ with $\tau_{i-1/2} + \beta_i + \tau_{i+1/2}$ i-handles for each i.*

Proof. By Corollary 5.3, there is a handle decomposition with no 0- or 1-handles. Similarly, we can dispense with $(n-1)$- and n-handles. This gives a chain-complex of free abelian groups whose homology is that of $H_*(W, \partial_- W)$. We put this chain-complex into normal form; then it is a direct sum of elementary subcomplexes, each with rank1 or 2, and differential either

$$0 \to \mathbb{Z} \to 0 \qquad \text{or} \qquad 0 \to \mathbb{Z} \xrightarrow{\sigma} \mathbb{Z} \to 0$$

Now the change of base needed to put the chain complex in normal form can be induced by a sequence of elementary automorphisms of the chain groups, and by Theorem 4.8, each of these can be induced by a change in handle decomposition. It remains only to remove the elementary subcomplexes with $\sigma = 1$. But Theorem 5.6 (extended by Remark 5.9) assures us that such pairs of handles may be cancelled. □

It does not seem to be easy to obtain a reasonable theorem of this kind without assuming simple-connectivity. The best known are Theorems 7.7 and 7.8 below .

Chapter 7

Simple Neighbourhoods

We shall now give a very brief discussion of Mazur's concept of simple neighbourhoods; however, we make no attempt to give complete proofs. The details would be comparatively trivial to supply if we were discussing combinatorial manifolds, so the reader may refer to think of these (nearly all the proofs in Part IV remain valid, only the details are much easier, on account of corners).

Let M^m be a compact manifold, K^k a finite complex. We call an imbedding f of K in M *tame* if M is covered by coordinate neighbourhood ($U_\alpha \subset M^m, \varphi_\alpha\colon U_\alpha \to \mathbb{R}^m$) such that each $\varphi_\alpha | f^{-1}(U)_\alpha\colon f^{-1}(U)_\alpha \to \mathbb{R}^m$ is linear on each simplex.

Definition 7.1. A submanifold U^m of $\mathring{M}^m$ is a *simple neighbourhood* of $f(K)$ if $K \subset \mathring{U}$, the inclusion $K \subset U$ is a simple homotopy equivalence, and $\pi_1(\partial U) \cong \pi_1(U \setminus K)$.

For example, if K is a submanifold, a tubular neighbourhood is a simple neighbourhood.

Proposition 7.2. *A smooth regular neighbourhood is a simple neighbourhood.*

Proof. This follows almost at once from the definitions; as to the last clause, we observe that if U is a regular neighbourhood, there is a map $\partial U \to K$ such that U is the mapping cylinder. So $U \setminus K \cong \partial U \times [0, 1)$. □

Proposition 7.3. *A smooth regular neighbourhood has a finite handle decomposition with one h^i corresponding to each simplex σ^i of K.*

Proof. By induction over simplexes of K; in fact, the handle is simply obtained by thickening the simplex. □

Remark 7.4. Conversely, any handle decomposition may be 'unthickened' to the cores of the handles to give a corresponding CW complex K.

Theorem 7.5. *(Simple Neighbourhood Theorem) Let $m \geqq 6$, codim $K \geqq 3$. Then if U_1, U_2 are simple neighbourhoods of K, there is a diffeotopy of M, constant near K and away from $U_1 \cup U_2$, which moves U_1 to U_2.*

Proof. Let U_0 e a smooth regular neighbourhood of K in $\mathring{U}_1 \cap \mathring{U}_2$. For $i = 1, 2$, we shall show that $U_i = U_0 \cup (\partial U_0 \times I)$: the result then follows at once. Since for $j = 1, 2, 0$, $\partial U_j \subset U_j \subset U_j \setminus K$ induces isomorphisms of fundamental groups, we can apply Theorem 6.8, and the result follows. □

Remark 7.6. Suppose the condition $\pi_1(\partial U) \cong \pi_1(U \setminus K)$ in the definition strengthened to state that $\partial U \subset U \setminus K$ is a homotopy equivalence. Then in the above, we could prove the closure of $U_i \setminus U_c$ an h-cobordism, but it would still not in general follow that it was an s-cobordism, if the codimension of K was 1 or 2. This does work, though, if $\pi_1(\partial U) \cong 0$, $\mathbb{Z}_2$ or $\mathbb{Z}$.

Theorem 7.7. *(Non-stable Neighbourhood Theorem) Let K, K_1 be finite CW complexes and $\theta\colon K \to K_1$ a simple homotopy equivalence. Suppose $U^n \to K$ by unthickening, and K, K_1 have dimension $\leqq n-3$, $n \geqq 6$. Then U^n has a handle decomposition which mimics the cell decomposition of K_1.*

Proof. By a theorem of Whitehead, (improved), we can go from K to K_1 by a sequence of "formal moves" of dimension $\leqq (n-2)$. (Note $n \geqq 6$). We can imitate each of these by a change in handle decomposition: an elementary expansion by introducing a complementary pair of handles (Theorem 4.7), and an elementary collapse by a handle cancellation (Theorem 4.5). For this last, we must check the necessary conditions. If the collapsed cells have dimensions 0, 1, we can use Proposition 4.1; if their dimensions are r, $r+1$ where $2 \leqq r \leqq n-3$, we observe that 5.7 is satisfied and that $\partial \tilde{e}_i^{r+1} = \tilde{e}_j^r$ and apply Lemma 5.8 (ii). If the dimensions are 1, 2, check that the attaching S^1 of the 2-handle is homotopic, hence isotopic, to a circle which meets the b-sphere of the 1-handle only once. In principle, this completes the proof. □

The same arguments lead also to

Theorem 7.8. *(Relative Non-stable Neighbourhood Theorem) Suppose W^n has a handle decomposition on V with no i-handles for $i > n-2$. Assume $\pi_1(\partial_+ V) \cong \pi_1(V) \cong \pi_1(W) \cong \pi_1(\partial_+ W)$, $n \geqq 6$. Let (X, V) be a CW pair with no i-cells outside V for $i > n-2$ and $f\colon X \to W$ a simple homotopy equivalence rel V. Then W has a handle decomposition based on V with cells corresponding to those of X mod V.*

This is stated in a very sharp form (I hope not too sharp to be true), and we shall not give the proof.

Part V

Cobordism: Geometric Theory

These notes continue

- Part 0 Analytical Foundations
- Part I Geometrical Foundations
- Part II Theorems of Transversality and General Position
- Part IV Theory of Handle Decompositions

originally issued at Cambridge. Parts II and IV are available on request to the Secretary, Department of Pure Mathematics, The University, Liverpool, 3. These were not prepared in close connection with a seminar, so the acknowledgments mainly due are to those who originally developed the ideas:

- primarily Thom, also Atiyah (for much of Chapters 5 and 6),
- Milnor (for demonstrating the variety which cobordism could encompass),
- Conner & Floyd (for much of Chapters 6 and 7)
- and Graeme Segal - who first obtained the results of Chapter 7 in their present generality.

I had originally intended to include a Chapter 8, on exact sequences of cobordism groups of knots: this is now omitted, but the reader may refer to the Bourbaki seminar (no. 280, 1964/5) by A. Haefliger, which gives the argument I had intended to use. Since our Part III has not yet (and may well never) be written, I will define it, too, by the references which seem to me to give the most coherent account of existing general methods (excluding surgery).

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Chapter 1

Types of cobordism

We have already said that when W is a compact manifold with ∂W a disjoint union of closed sets $\partial_- W \cup \partial_+ W$, W is called a *cobordism* of $\partial_- W$ and $\partial_+ W$, and $\partial_- W$, $\partial_+ W$ are called *cobordant*. This concept is of great generality, and there is a wide variety of possible generalisations and restrictions. Our policy here will be to indicate the different kinds of alteration that may be made to the definition, each in the simplest possible case: these may then afterwards be combined ad lib. We establish a convention that each type of cobordism relation is specified by a description of $\partial_- W$ which are relevant, and there is then a corresponding set of properties of the W which are permitted for the cobordism: these will be made precise in this chapter for each new idea, but the convention will be in force subsequently. Note that it is always essential that the manifolds be compact; otherwise the trivial relation $V = \partial(V \times [0,1))$ shows that everything is a boundary.

Oriented cobordism

We consider only oriented manifolds M. Then the cobordisms W must also be oriented. We call the oriented manifold W an *oriented cobordism* between the oriented manifolds $\partial_- W$ and $\partial_+ W$ if at each $x \in \partial_- W$ (resp. $\partial_+ W$), the orientation of $\partial_- W$ (resp. $\partial_+ W$) followed by the inward (esp. outward) normal at x induces the orientation of W. In terms of homology classes, this becomes

$$\partial_*[W] = [\partial_+ W] - [\partial_- W].$$

Cobordism with a given structure group

In the first instance, the tangent bundle τ_m of M has structure group $\mathbf{GL}_m(\mathbb{R})$: an orientation is a reduction of this group to $\mathbf{SL}_m(\mathbb{R})$. We generalise this now by letting G be any topological group (usually, but not always compact) and $\varphi\colon G \to \mathbf{GL}_m(\mathbb{R})$ a homomorphism. Then a *G-structure* on M is a reduction of the group of τ_m to G. If φ is the inclusion of a closed subgroup, we can define this as a cross-section of the bundle associated to τ_m with fibre $\mathbf{GL}_m(\mathbb{R})/G$. In general, we form the classifying space $B(G)$, so φ induces a vector bundle ξ_G over $B(G)$: then a reduction of the group of τ_m is a pair (e, f), where $f\colon M \to B(G)$ is a map and $e\colon \tau_m \to f^*\xi_G$ a bundle isomorphism: two reductions are *equivalent* if there is a reduction of the induced bundle over $M \times I$ which induces them at the two ends.

The natural definition of a cobordism W now demands a reduction of the structure group τ_W. However, $\tau_W|\partial W \cong \tau_{\partial W} \oplus \varepsilon^1$ (we shall use ε^r to denote the trivial vector bundle of dimension r), so the induced structure of the boundary is a reduction of the group of $\tau_{\partial W} \oplus \varepsilon^1$, rather than of $\tau_{\partial W}$ itself. We can base an adequate definition on this, noting only that a convention about the choice of isomorphisms of $\tau_{\partial W}$ on $\tau_{\partial W} \oplus \varepsilon^1$ is necessary: viz. that the positive vector ε^1 is to be identified with the inward normal to $\partial_- W$ in W, but with the outward normal on $\partial_+ W$ (this is necessary to obtain an equivalence relation: see below).

However, the most satisfactory general theory uses a further weakening of the concept, and some preliminary notation is necessary. Suppose given a commutative diagram of groups and homomorphisms

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & G_{n-1} & \xrightarrow{i_{n-1}} & G_n & \xrightarrow{i_n} & G_{n+1} & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n+1}} & \\
\cdots \longrightarrow & \mathbf{GL}_{n-1}(\mathbb{R}) & \longrightarrow & \mathbf{GL}_n(\mathbb{R}) & \longrightarrow & \mathbf{GL}_{n+1}(\mathbb{R}) & \longrightarrow \cdots
\end{array}$$

(in the lower row we have the natural inclusions); then we say we have a *stable group G*. A *weak G-structure* on M is prescribed by choosing an integer r and reduction (e, f) of the group of $\tau_m \oplus \varepsilon^r$ to G_{m+r}; (r, e, f) and (r', e', f') are *equivalent* if the reductions (e, f) and (e', f') of $\tau_m \oplus \varepsilon^s$ are so for some $s \geqq r, r'$. Then if a cobordism W has a weak G-structure, it induces weak G-structures on $\partial_- W$, $\partial_+ W$ (using the above convention to identify $\tau_{\partial W}$ with $\tau_{\partial W} \oplus \varepsilon^1$): we call it a cobordism between these manifolds with the induced structures.

Cobordism of Pairs

Let M be a submanifold of N; then we call (N, M) a pair. If (W, V) is a pair of manifolds with boundary, and W is a cobordism of $\partial_- W$ to $\partial_+ W$, we set $\partial_- V = V \cap \partial_- W$, $\partial_+ V = V \cap \partial_+ W$. Our definition of submanifold then implies that V is a cobordism of $\partial_- V$ to $\partial_+ V$, and we shall call the pair (V, W) a cobordisms of the pair $(\partial_- W, \partial_- V)$ to the pair $(\partial_+ W, \partial_+ V)$.

We can also restrict the structure groups of the tangent bundles of W and V separately; more important, however, is to consider the normal bundle of V in W, which has group $\mathbf{GL}_q(\mathbb{R})$ if q is the codimension of V in W. Given $\varphi_q\colon G_q \to \mathbf{GL}_q(\mathbb{R})$, a reduction to G_q of the group of the normal bundle of V in W can be called a *normal G_q-structure* of V in W: it induces normal G_q-structures of $\partial_- V$ in $\partial_- W$ and $\partial_+ V$ in $\partial_+ W$. Note that here there is no need to speak of weak structures and of identifications: the definition is more natural than the one above, and we have the notion of cobordism of pairs with normal G_q-structure.

Cobordism of Maps.

A map means a continuous (or, if preferred, smooth) map $f\colon M^m \to N^n$ of compact smooth manifolds. If V and W are cobordisms, and $F\colon V \to W$ defines by restriction maps $F_-\colon \partial_- V \to \partial_- W$ and $F_+\colon \partial_+ V \to \partial_+ W$, we call F a cobordism of F_- to F_+. In particular, a homotopy of f is a cobordism. Since (c.f. proof of 0, 2.2) every map is homotopic to a smooth map, and homotopic smooth maps are smoothly homotopic, the restriction to smooth maps f makes no difference. The special case when f is an imbedding gives cobordism of pairs above: we could also restrict f to be an immersion. Also since (II, 5.3) if $n > 2m$, any map is homotopic to an imbedding, and if $n > 2m+1$, homotopic imbeddings are isotopic, all these theories agree in such a stable range. It is also possible to replace f by an imbedding in $N \times \mathbb{R}^{2m}$, and restrict the group of the normal bundle.

Cobordism of Bounded Manifolds.

Let W be a manifold with corner; suppose the closed parts into which $\angle W$ divides ∂W are separated into three, with disjoint interiors: $\partial_- W$, $\partial_c W$ and $\partial_+ W$, where $\partial_- W$ and $\partial_+ W$ are disjoint, and the manifolds $\partial_- W \cup \partial_+ W$ and $\partial_c W$ have common boundary $\angle W$. Then we call W a cobordism of $\partial_- W$ to $\partial_+ W$. We also write $\angle_- W = \angle W \cap \partial_- W = \partial\partial_- W$ and $\angle_+ W = \angle W \cap \partial_+ W = \partial\partial_+ W$, so $\partial_c W$ is a cobordism of $\angle_- W$ to $\angle_+ W$.

By itself, this definition gives nothing: any manifold M with boundary is cobordant to the empty set by the manifold W obtained from $M \times I$ by rounding corners at $M \times I$. So the interesting cases are those in which an extra condition is imposed on the cobordism $\partial_c W$.

Cobordism with a cohomology class; bordism

First consider pairs (M, α) with $\alpha \in H^r(M; G)$. Then (W, α) is a cobordism of $(\partial_- W, \beta)$ to $(\partial_+ W, \gamma)$ if α restricts to β on $\partial_- W$ and to γ on $\partial_+ W$. Now the functor $H^r(-; G)$ is representable by the Eilenberg-MacLane complex $K = K(G, r)$, so we can equally regard α as a homotopy class of maps $M \to K$.

More generally, let K be any space and consider pairs (M, α) where α is a map $M \to K$. The definition of cobordism is the same as above. Note that if $\delta\colon M \times I \to K$ is a homotopy of α to α', $(M \times I, \delta)$ can be regarded as a cobordism of (M, α) to (M, α'). We shall later consider the dependence of the notion on K (rather than M), and will then say that (M, α) determines a *bordism class* of K.

If L is a subcomplex L, we shall also consider the cobordism relation for manifolds with boundary, where (M, α) is a pair consisting of a manifold M with boundary and a map of pairs $\alpha\colon (M, \alpha) \to (K, L)$: a cobordism will be a pair (W, β) where $\beta\colon (W, \partial_c W) \to (K, L)$ restricts to the given map of $(\partial_- W, \angle_- W)$ and of $(\partial_+ W, \angle_+ W)$.

Equivariant cobordism

Let G be a Lie group, which it is convenient to assume compact. We consider pairs (M, φ) where M is a manifold and $\varphi\colon M \times G \to M$ defines a smooth action of G on M. This induces a G-action on ∂M. then if W is a cobordism, with G-action φ, (W, φ) is an equivariant cobordism of $(\partial_- W, \varphi_-)$ to $(\partial_+ W, \varphi_+)$ if φ_-, φ_+ are G-actions induced by φ. A variant is obtained if we restrict the isotropy groups of φ to lie in an assigned class of subgroups of G - for example, if we have fixed-point-free actions.

The remaining examples involve connectivity, and we will see in Chapter 2 that they are of a slightly different nature: we shall call them all '*of type (C)*'.

Connected cobordism

Here we consider only connected (hence, in particular, non-empty) manifolds M. The cobordisms are restricted by the requirement that $\partial_- W$, $\partial_+ W$ and W all be connected. A natural extension of this is

k-connected cobordism

We now require M to be k-connected, for some integer $k \geqq 1$. In this case, of course, M is orientable: we make the further convention that M is oriented, The corresponding kind of cobordism is an oriented cobordism W, will later play a special role. Also if $\dim M \leqq 2k$, then M must be a homotopy sphere.

h-cobordism

The natural way to fit this into our present context seems to be to fix a space X, and insist that each manifold M under consideration to be provided with a homotopy equivalence $h_M\colon M \to X$. A cobordism W must then satisfy the condition that h_W extends the maps $h_{\partial_- W}$ and $h_{\partial_+ W}$: this implies that the inclusion maps of $\partial_- W$, $\partial_+ W$ into W are homotopy equivalences. It is usually more convenient in this case also to restrict to oriented cobordism. The only case to be singled out later is when X is a sphere.

I-cobordism

Here, X is a fixed closed manifold, and we consider only pairs (M, h_M), where h_M is a diffeomorphism of M on X. A cobordism is a pair (W, h_W) where h_W is a diffeomorphism of W on $X \times I$ inducing the diffeomorphisms $h_{\partial_- W \times 0}$, $h_{\partial_+ W \times 1}$ on the boundary. Naturally, this again is a trivial theory which we will only use in conjunction with others: we usually indicate its application when X is a sphere S^n (the commonest case) by referring simply to 'cobordism of S^n'.

Concordance

X is a fixed simplicial complex, and we consider pairs (M, h_M) where $h_M : X \to M$ is a smooth triangulation of M by a (linear) subdivision of X. Cobordisms must be triangulated by $X \times I$. We shall not give the theory, nor full definitions for these notions, but mention them for completeness. The word 'concordance' is sometimes also used for I-cobordism.

Cobordism with a homotopy class

Consider pairs (M, α), where M^m is simply-connected (so that the base points are irrelevant) and $\alpha \in \pi_r(M)$. We call (W, α) a cobordism of $(\partial_- W, \beta)$ to $(\partial_+ W, \gamma)$ if the inclusion maps send β and γ to α. We can also replace homotopy by homology, or the sphere S^r by another space K (and in some cases weaken the requirement of simple-connectivity). We will later need the restriction $n - 2 \geqq r$ (or $\dim K$).

Theorem 1.1. *In all cases, the relation "M is cobordant to M'" is an equivalence relation.*

Definition 1.2. The equivalence classes are called *cobordism classes*.

Proof. Reflexivity We use $M \times I$ to provide a cobordism of M to itself. In each case, any additional structure on M automatically defines one on $M \times I$ which extends it: for the natural projection π on M is homotopy equivalence, so the homotopy conditions extend, $\pi^* \tau_M \oplus \varepsilon^1 = \tau_{M \times I}$, and a G-action on M defines one on $M \times I$ by the formula $(m, t).g = (m.g, t)$. That $M \times I$ provides a cobordism of M to itself is now trivial except in the case of cobordism with a structure group, there we must use our convention about orientation.

Symmetry Let W^w be a cobordism of $\partial_- W$ to $\partial_+ W$: we wish to interchange their roles to call it a cobordism of $\partial_+ W$ to $\partial_- W$. This again is trivial in every case except that of cobordism with a structure group, where we must change the (weak) G-structure of W to 'interchange the ends'. We change it by observing that one weak structure induces another, using the identification of $\tau_W \oplus \varepsilon^r$ with itself by reflection in one of the line bundles constituting ε^r. The induced structure on $\tau_{\partial W} \oplus \varepsilon^{r+1}$ differs from the desired one by reflections in two line bundles: the product of two reflections is homotopic to the identity, so the induced structure is equivalent to the one required.

Transitivity Let W_1, W_2 be cobordisms with $N_0 = \partial_- W_1$, $N_1 = \partial_+ W_1 = \partial_- W_2$, $N_2 = \partial_+ W_2$. To obtain a cobordism of N_0 to N_2, we will glue W_1 to W_2 along N_1 (c.f. I, 7). This works without difficulty for cobordism with a structure group (our convention is natural here) and for most of the others. In fact, we need only take care with the glueing for cobordism of pairs and for equivariant cobordism.

In the case of pairs, let $(N_1, M_1) = \partial_+(W_1, V_1) = \partial_-(W_2, V_2)$. We choose collar neighbourhoods of N_1 in W_1 and in W_2 which respect the submanifolds V_1 and V_2: this is possible by I, Theorem 3.15. If we now glue, $V_1 \cup V_2$ becomes a smooth submanifold.

For G-cobordism, we first observe that every G-manifold has an equivariant Riemannian structure, obtained by taking any such structure, looking at its transforms by element of G, and integrating with respect to Haar measure on G, (which is legitimate since the Riemannian structures form a convex subset of a Banach space). The construction of I, Proposition 3.8 now gives equivariant collar neighbourhoods of the boundary; glueing as in I, 7.2 we see that the action of G remains differentiable. The proof of the theorem is now complete. $\square$

Chapter 2

Cobordism groups and rings

We next investigate the various possible structures that can be put on the sets of cobordism classes: here the two key remarks are that disjoint union will (in most cases) define a sum operation making the set of classes an additive abelian group, and that Cartesian product induces various multiplicative structures. A few more delicate operations will be defined later on.

Lemma 2.1. *Disjoint union defines an addition which turns the set of cobordism classes (of a given dimension) into an abelian group,* except *for cobordism of type (C).*

Proof. The other kinds of structure pass at once to the disjoint union. Union is compatible with cobordism: if V, W are cobordisms of $\partial_- V$ to $\partial_+ V$, $\partial_- W$ to $\partial_+ W$, then the disjoint union V, W is a cobordism of $\partial_- V \cup \partial_- W$ to $\partial_+ V \cup \partial_+ W$. Thus we have a binary operation on the set of cobordism classes, which is commutative and associative since disjoint unions are. The empty manifold acts as zero.

We obtain an inverse to W whenever $M \times I$ may be regarded as a cobordism of the disjoint union $M \times 0 \cup M \times 1$ to the empty set (the induced structure on $M \times 0$ must coincide with that on M: on $M \times 1$ it can be different). This is immediate in each case except cobordism with a given structure group, where we have an orientation-reversal on $M \times 1$ (as in the proof of symmetry in Theorem 1.1).

In the case where connectivity is important, we will use connected sum instead of disjoint union. For h-cobordism and I-cobordism we need to take X as a sphere for this to give a group structure: in other cases it gives a map relating three different cobordism sets (of X_1, X_2 and $X_1 \sharp X_2$) - as indeed did disjoint union. □

Lemma 2.2. *In all cases except cobordism of maps and equivariant cobordism, connected sum of connected manifolds of dimension > 0 is a commutative associative operation with unit, compatible with cobordism. The set of equivalence* *classes thus acquires an abelian group structure, provided for h- and I- cobordisms we take X as a sphere. In cases where disjoint union and connected sum both define a group structure on cobordism classes, the two structures are the same.*

Proof. We first check that connected sum can be made compatible with all the extra structures except a G-action. If M_1 and M_2 are k-connected, so is $M_1 \sharp M_2$ - except in the trivial case when $\dim M_i \leqq k$ (so the M_i are contractible, and have boundaries). For h-cobordism, we have maps $h_i \colon M_i \to X_i$, where the X_i can be taken as manifolds. It is then simple to adjust the h_i by homotopies to respect the discs used to define $\sharp$, and thus obtain a homotopy equivalence $M_1 \sharp M_2 \to X_1 \sharp X_2$. The corresponding assertion for I-cobordism is trivial. Weak G-structures can be defined near the discs $f_i(D^m) \subset M_i$ by framings, induced by the f_i from the standard framing $e_1, \cdots, e_m$ of $\mathbb{R}^m$. If e_{m+1} is the extra basis element when we add a trivial line bundle to τ_m, we change the framing on D^m as follows: at $\nu \in D^m$, reflect in the hyperplane perpendicular to $e_{m+1} - \nu$, then in the one perpendicular to e_{m+1}. This can be achieved by a homotopy (D^m is contractible). If the new framing is $e'_1, \cdots, e'_{m+1}$, then e'_{m+1} is the inward normal vector to S^{m-1} in D^m. Thus the weak G-structures on $M_1 \setminus f_1(D^m)$ and $M_2 \setminus f_2(D^m)$ fit together along S^{m-1} after changing the sign of e'_{m+1}. For cobordism of connected pairs, we glue both manifolds simultaneously, using imbeddings $f_i \colon (D^n, D^m) \to (N, M)$ - the theory of this operation is essentially the same as for ordinary connected sum. With homotopy classes, we consider pairs $(M_i^m, \alpha_i), \alpha_i \in \pi_r(M_i)$. Here we need $r \leqq m-2$. Then α_i determines a homotopy class in $M_i^m \setminus \mathrm{Int} f_i(D^m)$, and hence in $M_1 \sharp M_2$: we add the resulting classes. With cohomology classes, we first adjust the maps $\alpha_i \colon M_i \to K$ by homotopies so that $\alpha_i f_i$ has image the base point: we then have a natural induced map of $M_1 \sharp M_2$.

In each of these cases, the operation is clearly commutative and associative, and the sphere S^m (associated to the weak framing induced by that of $\mathbb{R}^{m+1}$, and zero homotopy and cohomology classes) acts as unit - this needs a moment's thought in the case of a structural group.

We must next check that the operation is compatible with cobordism. First, there is a question of orientation: but if any condition on structural groups provides an orientation of the manifold, the connected sum is unique: if not (but the manifolds are still orientable) we can take a further connected sum with a non-orientable manifold, and orientation becomes irrelevant, and we then add the inverse of the manifold (see below).

[Note here the use of an earlier convention that if other conditions on a manifold require orientability, we add an orientation to the specifications]. Next, let V and W be connected cobordisms, of dimension $n+1$, and $f_- \colon D^n \to \partial_- V$, $f_+ \colon D^n \to \partial_+ V$, $g_- \colon D^n \to \partial_- V$, and $g_+ \colon D^n \to \partial_+ V$ be used to define the connected sums $\partial_- V \sharp \partial_- W$ and $\partial_+ V \sharp \partial_+ W$. As above, we suppose either that all manifolds are oriented or that V, W are non-orientable. Then we can join $f_-(0)$ to $f_+(0)$ by an arc α in V, and thicken to obtain an imbedding $F \colon D^n \times I \to V$ with $f_- = F|D^n \times 0$ and $f_+ = F|D^n \times 1$ (if orientations do note fit at the first attempt, V is by hypothesis non-orientable, and we compose the homotopy class of α with and orientation-reversing loop). The hypothesis $n \geqq 2$ is needed to use general position to get the arc imbedded, but if $n = 1$, a more direct construction suffices. Similarly define $G\colon D^n \times I \to W$. Now delete the interiors of the images of F and G and glue the boundaries, and we have a cobordism of $\partial_- V \sharp \partial_- W$ to $\partial_+ V \sharp \partial_+ W$. the verification that this construction is compatible with extra structure is the same as for $\sharp$ itself, except in the case of simple-connectivity. Here, if $n > 2$, general position shows that the complement of an arc in V is simply-connected if V is, so we consider only the case $n = 2$. The only simply-connected closed 2-manifold is S^2, and if $S^2 = \partial_- V$, observe that $\pi_1(V \setminus \alpha)$ is generated by conjugates of a loop encircling α, which can be taken in S^2, but is then already null-homotopic in $S^2 \setminus f_-(D^2)$ (a contractible set). So $V \setminus \alpha$ is simply-connected in this case also.

It remains to obtain inverses. Note that S^n bounds D^{n+1}, and the zero structures on S^n all extend to D^{n+1}. Conversely, let W be a connected cobordism with $\partial_+ W = \varnothing$. Then we assert that $\partial_- W$ is cobordant to the zero class. For deleting the interior of an imbedded disc from W, we obtain a W' with $\partial_- W = \partial_- W'$, $\partial_+ W' = S^n$. The verification that a structure on W induces one on W' is again the same as for $\sharp$, and the induced structure on S^n extends to D^{n+1}, hence is the zero structure.

The inverse is now obtained by change of orientation, as usual: say M gives rise to M'; together they bound $M \times I$. Now take an imbedding $f\colon D^m \to M$: this extends to $f \times 1\colon D^m \times I \to M \times I$. Delete the interior of the image and round the corners: this gives W with $\partial W = M \sharp M'$. But now any structure on M induces one on $M \times I$ and (again by the same argument as for $\sharp$) on W. Thus M' is indeed inverse to M. Note that in the cases of I- and h- cobordism with $X = S^n$, this construction gives W diffeomorphic (resp. homotopy equivalent) to D^{n+1}: then deleting an imbedded disc gives W' diffeomorphic (or homotopy equivalent) to $S^n \times I$.

The last assertion of the lemma is checked by constructing a cobordism of $M \cup M'$ to $M \sharp M'$: for this we take $M \times I \cup M' \times I$ and attach a 1-handle to join $M \times 1$ and $M' \times 1$. Then for a structure group, a pair, or a cohomology class (the three remaining cases), we note that we can use a framing over the added handle, add a handle-pair (with trivial normal bundle), or map the handle to the base point.

This concludes our discussion of additive structure. There is less to say about multiplicative structure in general which is not obvious: the general rule is that the natural ('external') product is a little too precise, and we must weaken its induced structure to obtain a more useful multiplication.

First, products are compatible with cobordism: if W is a cobordism from $\partial_- W$ to $\partial_+ W$, then $W \times M$ is a cobordism from $\partial_- W \times M$ to $\partial_+ W \times M$. Also, products are associative, and distributive over disjoint union (though not over connected sum), and there is a natural diffeomorphism of $M' \times M$ on $M \times M'$, which gives rise in most cases to some sort of commutativity of multiplication.

Next, let us examine cases in a little more detail. If M_1, M_2 have weak G_1 resp. G_2 structures, then $M_1 \times M_2$ has a natural induced weak $G_1 \times G_2$-structure, and hence also G_3-structure if we have a morphism $\psi\colon G_1 \times G_2 \to G_3$ (see next chapter for definitions). This includes oriented cobordism, for example.

If (M_1, V_1) and (M_2, V_2) are pairs, the natural product is a set of 4 manifolds. Here, the most useful notion is to multiply a pair (M, V) by a manifold M'. Note that the group of the normal bundle is unaltered.

Again, it is unwise to multiply two manifolds with boundary - the resulting structure is so complicated - and it is more likely to be profitable to multiply manifolds with boundary by closed manifolds.

Equivariant cobordism has a natural external product: actions of G on M and of H on N induce an action of $G \times H$ on $M \times N$ hence of any subgroup. If, in particular, $G = H$, we have a diagonal action of G.

The one remaining case (since we exclude connected cobordism, after our failure to obtain distributivity) is cobordism with a cohomology class. Given two pairs (M_1, α_1) and (M_2, α_2), where $\alpha_i\colon M_i \to X_i$, the exterior product is the pair $(M_1 \times M_2, \alpha_1 \times \alpha_2)$. We will usually have a map $f\colon X_1 \times X_2 \to X_3$, and replace $\alpha_1 \times \alpha_2$ by $f \circ (\alpha_1 \times \alpha_2)$, to obtain a bordism class of X_3. □

This seems the most appropriate place to mention a general method of constructing exact sequences, several illustrations of which will appear later. Here we will not be too precise.

Suppose two kinds of structure specified, called an α-structure and a β-structure, with the latter stronger than the former. For example, we may consider structure groups G_1 and $G_2 \subset G_1$, or maps to spaces X_1 and $X_2 \subset X_1$, or actions of groups H_1 and $H_2 \supset H_1$, or k_1-connectivity and $k_2 (> k_1)$-connectivity.

By Ω_n^α and Ω_n^β we denote the cobordism groups of manifolds with α- (resp. β-) structure; and by $\Omega_n^{\alpha,\beta}$ the cobordism group of bounded manifolds with α-structure, whose boundaries have a β-structure including the given α-structure.

Lemma 2.3. *There is an exact sequence*

$$\cdots \longrightarrow \Omega_n^\beta \longrightarrow \Omega_n^\alpha \longrightarrow \Omega_n^{\alpha,\beta} \longrightarrow \Omega_{n-1}^\beta \longrightarrow \Omega_{n-1}^\alpha \longrightarrow \cdots$$

Proof. (sketch) The first two maps are the obvious ones; the third is induced by taking the boundary. Exactness at Ω_{n-1}^β is immediate. It is clear that the composite of two maps in the sequence is zero. If M is bounded, and ∂M (as a β-manifold) bounds V, we can glue M to V along ∂M to obtain a closed manifold M' with α-structure. A cobordism W of M' to M is obtained from $M' \times I$ by introducing a corner at $\partial M \times 0$, and setting $\partial_- W = M \times 0$, $\partial_c W = V \times 0$ and $\partial_- W = M' \times 1$. Finally, if the closed α-manifold M is trivial as bounded (α, β)-manifold, the corresponding cobordism W has $\partial_- W = M$, $\partial_+ W = \varnothing$, and so $\partial_c W$ a closed β-manifold, α-cobordant to M.

In some cases, any manifold with α-structure then has a β-structure except on a closed subcomplex or submanifold. Then $\Omega_n^{\alpha,\beta}$ can be calculated differently, for if M is a bounded (α, β)-manifold, $K \subset \operatorname{Int} M$ the exceptional subcomplex and L a 'smooth regular neighbourhood' or tubular neighbourhood of K, then

M is (α, β)-cobordant to L by W, obtained from $M \times I$ by rounding the corner at $\partial M \times 1$ and introducing one at $\partial L \times 1$. An analogous remark applies to cobordisms. $\square$

Chapter 3

Examples

Before we proceed with the theory, we give here a number of examples which show how the different variants on the simple cobordism relations, as listed in Chapter 1, may be combined in useful ways. We also take the opportunity of introducing the notation for those groups to which we will refer later, and of making clear the application of the results of Chapter 2 to the cases which arise (though we shall not repeat the proofs).

The simplest case of all is unrestricted cobordism of closed n-manifolds, We obtain a group, classically denoted by $\mathfrak{N}_n$: but which (to fit into a systematic notation) we shall write as $\Omega_n^{\mathbf{O}}$. Multiplication gives a commutative and associative product $\Omega_m^{\mathbf{O}} \times \Omega_n^{\mathbf{O}} \to \Omega_{m+n}^{\mathbf{O}}$, and a point acts as unit. We thus have a commutative graded ring $\Omega_*^{\mathbf{O}}$. Each element is its own additive inverse, so we can consider $\Omega_*^{\mathbf{O}}$ as an algebra over $\mathbb{Z}_2$.

Next we have oriented cobordism, giving a group $\Omega_n^{\mathbf{SO}}$ (formerly written Ω_n). Multiplication gives a graded ring, as before, which is commutative in the graded sense, and has a unit: we write $\Omega_*^{\mathbf{SO}}$.

More generally, let G be any stable group. We consider the cobordism groups of manifolds with weak G-structure on the stable tangent bundle - say provisionally $\Omega_n'^G$. Then we can obtain a bilinear product $\Omega_m'^G \times \Omega_n'^G \to \Omega_{m+n}'^G$ by imposing on G the axiom.

(M) We have a family of maps $\psi_{m,n}\colon G_m \times G_n \to G_{m+1}$ such that following diagrams commute up to conjugating by an element in the component of the identity:

$$\begin{array}{ccc} G_m \times G_n & \xrightarrow{\psi_{m,n}} & G_{m+n} \\ \downarrow{\scriptstyle i_m \times 1} & & \downarrow{\scriptstyle i_{m+n}} \\ G_{m+1} \times G_n & \xrightarrow[\psi_{m+1,n}]{} & G_{m+n+1} \end{array}$$

$$\begin{array}{ccc}
G_m \times G_n & \xrightarrow{\psi_{m,n}} & G_{m+n} \\
\Big\downarrow{\scriptstyle 1\times i_n} & & \Big\downarrow{\scriptstyle i_{m+n}} \\
G_m \times G_{n+1} & \xrightarrow[\psi_{m,n+1}]{} & G_{m+n+1}
\end{array}$$

$$\begin{array}{ccc}
G_m \times G_n & \xrightarrow{\psi_{m,n}} & G_{m+n} \\
\Big\downarrow{\scriptstyle \varphi_m\times\varphi_n} & & \Big\downarrow{\scriptstyle \varphi_{m+n}} \\
\mathbf{GL}_m(\mathbb{R}) \times \mathbf{GL}_n(\mathbb{R}) & \longrightarrow & \mathbf{GL}_{m+n}(\mathbb{R})
\end{array}$$

The product becomes associative if we impose also the axiom

(A) The following diagram also commutes (in the same sense)

$$\begin{array}{ccc}
G_l \times G_m \times G_n & \xrightarrow{\psi_{l,m}\times 1} & G_{l+m} \times G_n \\
\Big\downarrow{\scriptstyle 1\times\psi_{m,n}} & & \Big\downarrow{\scriptstyle \psi_{l+m,n}} \\
G_l \times G_{m+n} & \xrightarrow{\psi_{l,m+n}} & G_{l+m+n}.
\end{array}$$

The product gives a commutative graded ring if we insist also the commutativity on the diagram

(C)

$$\begin{array}{ccc}
G_m \times G_n & \xrightarrow{\varphi_{m+n}\psi_{m,n}} & \mathbf{GL}_{m+n}(\mathbb{R}) \\
\Big\downarrow{\scriptstyle T} & & \Big\downarrow{\scriptstyle T'} \\
G_n \times G_m & \xrightarrow{\varphi_{m+n}\psi_{m,n}} & \mathbf{GL}_{m+n}(\mathbb{R})
\end{array}$$

where T is the natural interchange of factors, and T' means conjugation by an element whose determinant has sign $(-1)^{mn}$.

We shall also need a stability axiom

(S) There is a function q_n of n, increasing (in the weak sense) and tending to infinity, such that i_n is q_n-connected.

We now show that if (**S**) holds, we can replace the structure group on the stable tangent bundle (which has been a constant source of difficulty up to this point) by a structure group on a normal bundle.

Lemma 3.1. *Suppose given a commutative diagram*

$$\begin{array}{ccc}
G_r \times G_s & \xrightarrow{\psi} & G_{r+s} \\
\Big\downarrow{\scriptstyle \varphi_r\times\varphi_s} & & \Big\downarrow{\scriptstyle \varphi_{r+s}} \\
\mathbf{GL}_r \times \mathbf{GL}_s(\mathbb{R}) & \longrightarrow & \mathbf{GL}_{r+s}(\mathbb{R})
\end{array}$$

such that the map $\psi_o : G_r \to G_{r+s}$ induced by ψ is c-connected. Let K be a CW complex of dimension $\leqq \min(c, r-2)$, and ξ^r, η^s vector bundles over K, with a G_s-structure on η^s. Then the function f induce by ψ from G_r-structures on ξ^r to G_{r+s}-structures on $\xi^r \oplus \eta^s$ is bijective.

Proof. Let X_i be the classifying space for $G_i (i = r, s, r+s)$; E_i the total space of the principal bundle with fibre $\mathbf{GL}_i(\mathbb{R})$ induced over X_i by φ_i. Write E_ξ, E_η, $E_{\xi\oplus\eta}$ for the spaces of the corresponding principal bundles over K. Then G_r-structures of ξ correspond to sections of the bundle over K with total space $E_\xi \times_{\mathbf{GL}_r(\mathbb{R})} E_r$; similarly for $\xi \oplus \eta$. But the G_s-structures of η induces a fibrewise map

$$E_\xi \times_{\mathbf{GL}_r(\mathbb{R})} E_r \to E_{\xi\oplus\eta} \times_{\mathbf{GL}_{r+s}(\mathbb{R})} E_{r+s} \tag{3.2}$$

and the induced map of fibres is $E_r \to E_{r+s}$, which is at least $\min(c+1, r-1)$-connected since $X_r \to X_{r+s}$ is $(c+1)$-connected and $\mathbf{GL}_r(\mathbb{R}) \to \mathbf{GL}_{r+s}(\mathbb{R})$ is $(r-1)$-connected. Thus (3.2) is at least $(1 + \dim K)$- connected, so any map of K to the second term can be factorised (up to homotopy) through the first, and f is surjective; moreover, the result is unique up to homotopy, so f is bijective. (We use the well-known result - a corollary of the homotopy lifting theorem - that sections of a bundle are homotopic only if they are homotopic through sections). □

Corollary 3.3. *Let $M^m \subset \mathbb{R}^{m+N}$ have a weak G-structure, where the stable group G satisfies* (**M**), (**A**) *and* (**S**), *and $q_N \geqq m$. Then the normal bundle has a G_N-structure; conversely, this implies a weak G-structure on M.*

Proof. In this case, $\xi \oplus \eta$ has a standard framing, and hence G-structure. We use (**A**) only to identify the ψ_0 of the lemma with a composite of maps i_n. □

Corollary 3.4. *If G satisfies* (**M**), (**A**) *and* (**S**), *and $N \geqq m+2$, $q_N \geqq m+1$, then Ω'^G_m is isomorphic to the cobordism group of pairs (S^{m+N}, N^m), with G_N as group of the normal bundle.*

Strictly speaking, this uses the extension of the lemma where we fix a G-structure of the restriction of ξ to a subcomplex of K: the proof is the same. It is more convenient to use normal than tangent bundles; accordingly, by Ω^G_m we will denote the cobordism group of m-manifolds with a G-structure on the stable normal bundle. By (3.4), under (**M**), (**A**) and (**S**) we have $\Omega'^G_m = \Omega^G_m$.

Let us observe, before leaving our general discussion of cobordism with a structure groups, that if the $\varphi_r(G_r)$ are not all contained in the identity components of the groups $\mathbf{GL}_r(\mathbb{R})$, then the 'orientation reversal' used in Lemma 2.1 to define inverses does not in fact change the G-structure: up to homotopy, we can realise it by conjugating by an element of G. In this case, we call G *non-orientable*, and observe that Ω^G_* can be considered as a $\mathbb{Z}_2$-module. Otherwise, we call G *orientable*; then the class of a point in $\Omega^G_\mathbf{O}$ clearly has infinite order.

The important examples of stable groups G are the classical groups **O**, **SO**, **Spin**, **U**, **SU** and **Sp**, and the trivial group $\{1\}$. Of interest also are the groups **Spin**^{c}, **Pin**, and **Pin**^{c} of Atiyah, Bott and Shapiro (Topology 3 supp. 1; see esp. pp 7-10). Clearly, there are many ways of inventing more: for example, we can take products of the above with each other or with any group of linear operators on a finite dimensional vector space.

We next consider pairs (V^{m+q}, M^m), where V has a weak G-structure and the normal bundle an H_q-structure. We introduce no notation for this, since the cobordism problem here can be reduced to the previous case. More generally, consider the situation $M^m \subset V^{m+q} \subset S^{m+q+r}$, where the structure groups of the normal bundles are H_q and G_r. Then the normal bundle $M^m \subset S^{m+q+r}$ has an $H_q \times G_r$-structure.

We shall only consider the stable case $r > m + q + 1$ where the imbedding of V in S is irrelevant (we can always find one, and any two such are isotopic, by (II, 5.3)), though this restriction could be somewhat weakened.

Lemma 3.5. *Suppose H_q compact. Then the pair (V^{m+q}, M^m) is (G_r, H_q)-cobordant to the empty pair if and only if V^{m+q} is G_r-cobordant to zero and $M^m)$ is $G_r \times H_q$-cobordant to zero.*

Proof. The necessity of the condition is evident. We shall prove sufficiency by establishing a principle of 'extension of cobordism' (c.f. homotopy extension) which will frequently be of use when considering cobordism of pairs with various restrictions. In this case, we need a construction to extend a $G_r \times H_q$-cobordism of M^m to the empty set to a $(G_r \times H_q)$-cobordism of (V, M) to a pair (V', ψ). Since cobordism is an equivalence relation, it follows that V' is G_r-cobordant to φ, say by W'; then (W', φ) is the required (G_r, H_q)-cobordism of (V', ψ) to (ψ, φ).

Now since H_q is compact, we can suppose that it acts orthogonally on $\mathbb{R}^q$. Let N^{m+1} be the given $G_r \times H_q$-cobordism of M to φ: then there is an induced bundle over N with fibre D^q, whose total space we denote by L^{m+q+1}. Note that the restriction to M of this bundle is the normal bundle of M in V; hence we can identify a tubular neighbourhood of M in V with part of the boundary of L. We form $V \times I$, and attach L to $V \times 1$ by this identification, giving W. Since L and $V \times I$ have G_r-structures, which agree (by hypothesis, N is a cobordism of M with the $G_r \times H_q$ structure induced from V) on the par identified, W^{m+q+1} has a G_r-structure. Also, $M \times I \cup N = N'$ is a submanifold whose normal bundle has group H_q.

Set $V \times 0 = \partial_- W$. Then (W, N') is a (G_r, H_q) cobordism, and $N' \cap \partial_+ W = \varnothing$. This completes the proof of the lemma. □

Corollary 3.6. *The cobordism group of pairs (V^{m+q}, M^m), where V has a weak G-structure and the normal bundle an H_q-structure (H_q compact) is isomorphic to $\Omega^G_{m+q} \oplus \Omega^{G \times H_q}_m$.*

Proof. We have defined a map to the direct sum, and proved it a monomorphism; it clearly respects additive structure. Them map to $\Omega^{G \times H_q}_m$ is onto, for given a $(G \times H_q)$-manifold M^m, we construct as above a bundle over M with fibre D^q, and can take V as the double of this manifold. Finally, the image contains $\Omega^G_{m+q} \oplus 0$: we need only consider pairs with M empty.

We observed in Chapter 2 that the collection of the above cobordism groups (with m varying) was an Ω_*^G module if G satisfied all the axioms. The module structure clearly respects the direct sum splitting: thus $\Omega_*^{G\times H_q}$ is an Ω_*^G module - as is indeed clear directly. Note that if G is a stable group [satisfying (**S**)], then so is $G \times H_q$. □

Next, consider bordism: we denote the cobordism groups of manifolds M^m with weak G-structure and a map $M \to X$ by $\Omega_*^G(X)$: thus $\Omega_*^G \cong \Omega_*^G(point)$. If M has a boundary, (X, Y) is a pair, and we have a map $(M, \partial M) \to (X, Y)$, we obtain a group $\Omega_*^G(X, Y)$. It is also possible to use a group other than G (but mapping into G) for structure group of $\tau_{\partial M}$: this extension appears less interesting. If X has a base point $*$, the natural maps $* \to X \to *$ induce $\Omega_*^G \to \Omega_*^G(X) \to \Omega_*^G$ which split $\Omega_*^G(X)$ as a direct sum $\Omega_*^G \oplus \tilde{\Omega}_*^G(X)$. We will consider bordism in more detail in Chapter 5.

For equivariant bordism, we let H be a compact group of operators, and A a family of subgroups of H; G will continue to denote a stable group. Then the cobordism group of manifolds with G-structure and an action of H such that each isotropy group belongs to A will be denoted by $I_*^G(H; A)$. Note here that H must act on the G-structure. Since every isotropy group is necessarily closed, and since if a given subgroup of H occurs as isotropy group, then so do all its conjugates, we may always suppose the A is a union of conjugacy classes of closed subgroups. Equivariant cobordism will be studied in Chapter 7.

As to connected cobordism, we observe that already in Lemma 2.2. We proved that disjoint union was cobordant to connected sum, so that in dimensions $\geqq 1$, the connected cobordism group maps onto the usual one. The map is in fact bijective, since if W is a cobordism to $\varnothing$ of a connected manifold $\partial_- W$, then so is the component W' of W which contains $\partial_- W$. There are analogous results for k-connected cobordism, but we postpone these until the section on surgery (Part VI).[1]

By Lemma 2.2, H-cobordism class of homotopy n-spheres form a group: we denote it by Θ_n. Consider pairs (T^{n+q}, T^n), with a diffeomorphism $T^{n+q} \to S^{n+q}$ and a homotopy equivalence $T^n \to S^n$: we obtain another group Θ_n^q. If we frame the normal bundle also, we have a group $F\Theta_n^q$. If we replace the homotopy equivalence $T^n \to S^n$ by a diffeomorphism (I-cobordism of pairs), we get a group C_n^q: if we also have a framing, we obtain FC_n^q. If it is replaced by a smooth triangulation by a (linearly subdivided) simplex boundary, we get groups Γ_n, Γ_n^q, $F\Gamma_n$ and $F\Gamma_n^q$.

Further groups are obtained by making strong restrictions on the boundary. For example, call a manifold M^m *almost-closed* if a homotopy equivalence $h_{\partial M}\colon \partial M \to S^{m-1}$ is given. The corresponding kind of cobordism is that in which $\partial_c W$ is an h-cobordism. We write P_m for the cobordism groups of framed, almost-closed m-manifolds; P_m^q for the group of pairs (S^{m+q-1}, M^m), with framed normal bundle and ∂M almost-closed, and DP_m^q for the group of pairs (D^{m+q}, M^m) with the same restrictions (here, M^m is a submanifold of

[1] Part VI has not been written.

D^{m+q} but for P^q_m, M^m was a submanifold with boundary of S^{m+q-1}). Chapter 8 was to have given exact sequences which relate these groups of structures on spheres, but again we postpone fuller discussion until Part VI.[2]

To illustrate the generality of the definition in Chapter 1 we point out that the ordinary homotopy groups appear as a special case of cobordism groups: more precisely, $\pi_n(X)$ is the group of I-bordism classes of maps $S^n \to X$: our definition of the equivalence relation, and of addition, coincides with one of the traditional definitions.

We give no examples of cobordism with a homotopy class: no research seems to have been done in this direction.

[2]Part VI has not been written.

Chapter 4

Thom theory

Let ξ be a vector bundle with total space E_ξ and base B_ξ. If we assume at least that ξ is numerable (i.e., that there is a partition of unity subordinate to an open covering over each set of which ξ is trivial), then the structure group of ξ can be reduced, essentially uniquely, to the orthogonal group. We then define the *Thom space* of ξ, denoted by T_ξ, by taking the subspace A_ξ of E_ξ of all vectors of length $\leqq 1$, and identifying to a point (denoted ∞) the set $\dot{A}_\xi$ of vectors of length 1. We note that if B_ξ is a finite CW complex, so is T_ξ; if B_ξ is a smooth manifold, we can give ξ the structure of smooth vector bundle, and E_ξ and $T_\xi \setminus \{\infty\}$ then also acquire the structure of smooth manifolds. If B_ξ is compact, we can give an alternative description of T_ξ as the one-point compactification of E_ξ: the equivalence of this with the above follows from the observation that E_ξ is homeomorphic to the subbundle of vectors of length < 1.

Now let M_m be a submanifold of the compact manifold (perhaps with boundary) V^{m+q}, ξ the normal bundle. Then we can find an imbedding $h\colon A_\xi \to V$ defining a tubular neighbourhood of M in V (I, Theorem 3.14). If we now take V, and shrink to a point the complement of $\mathrm{Int}h(A_\xi)$, we obtain a space, and h defines a homeomorphism of T_ξ onto that space: thus we have an induced map $V \to T_\xi$. This is a preliminary version of the Thom construction.

Next, let $B(G)$ be a classifying space for G, where G is a topological group of orthogonal operators on $\mathbb{R}^q$, let $\omega(G)\colon E(G) \to B(G)$ be the universal bundle with fibre $\mathbb{R}^q$, having subbundles $A(G)$ with fibre D^q and $\dot{A}(G)$ with fibre S^{q-1}. We denote the Thom space by $T(G)$. In the sequel, we wish to be able to consider $B(G)$ as a smooth manifold, hence must weaken the requirement to being N-classifying, for some large enough integer N. Thus we can first replace the original $B(G)$ (given - say - by Milnor's construction) by the $(N+1)$-skeleton of its singular complex; next provided the homotopy groups of $B(G)$ (or equivalently of G) are countable, by a countable $(N+1)$-simplicial complex; then by a locally finite one, and finally imbed this last properly in Euclidean $(2N+3)$-space and take an open neighbourhood of which it is a deformation retract. More simply, if G is a compact Lie group (the only case of importance in the sequel), we use the orbit space under the diagonal action of G on the join of $(n+1)$ copies of itself.

Finally, given a pair (V^{m+q}, M^m) of compact manifolds, and a reduction to G of the group of the normal bundle ξ, we can find a bundle map $\xi \to \omega(G)$, which induces a map $(A_\xi, \dot{A}_\xi) \to (A(G), \dot{A}(G))$ and hence $T_\xi \to T(G)$. We say that the composite map $V^{m+q} \to T_\xi \to T(G)$ is obtained by the *Thom construction*.

The first significant result in cobordism theory is that the construction can, in a sense, be reversed.

Theorem 4.1. *Let G have countable homotopy groups. Then the Thom construction induces a bijective map of the set of cobordism classes of pairs (V^{m+q}, M^m), with V fixed and G as structure group of the normal bundle, onto the set of homotopy classes $[V : T(G)]$. If V is a sphere, the map is a group isomorphism.*

Proof. We must first show that the map is well-defined. Let $(V \times I, N^{m+1})$ be a cobordism of the appropriated kind, and suppose the construction already performed on the pairs $(V \times 0, \partial_- N)$ and $(V \times 1, \partial_+ N)$. It follows easily form the tubular neighbourhood theorem (I, 5.10) that the chosen tubular neighbourhoods of $\partial_- N$ and $\partial_+ N$ can be extended to one of N in $V \times I$, from which we can construct a map $V \times I \to T_\nu$ (ν the normal bundle of N in $V \times I$), extending the given maps on $V \times 0$ and $V \times 1$. Again, by the homotopy extension theorem, we can find a bundle map $\nu \to \omega(G)$ extending the chosen maps over ∂N. The composite $V \times I \to T_\nu \to T(G)$ is now a homotopy between the given maps $V \to T(G)$. Hence we have a well-defined mapping of the cobordism set into the homotopy set.

We next prove the map is onto. Since G has countable homotopy groups, we can suppose $B(G)$ a smooth manifold, and hence also $T(G) \setminus \{\infty\}$. We identify $B(G)$ with the zero cross-section in $E(G)$, and hence with a smooth submanifold, closed in $T(G)$. Observe that when we perform the Thom construction on (V^{m+q}, M^m) to obtain a map $f : V \to T(G)$, we have $f^{-1}(B(G)) = M^m$, since the construction is induced from a bundle map $A_\xi \to A(G)$. Now, conversely, suppose given a map $f : V \to T(G)$. By (II, 4.6), we can approximate f by $g : V \to T(G)$, transverse to the submanifold $B(G)$: the fact that ∞ is a singular point of $T(G)$ is irrelevant, since we can take $g = f$ in a neighbourhood of $f^{-1}(\infty)$, by (II, 4.7). If the approximation is close enough, $g \simeq f$. Since g is transverse, by (II, 4.2), $M^m = g^{-1}(B(G))$ is a submanifold of V^{m+q}. Also, by the definition of transversality, g induces a bundle map of the normal bundle ξ to M in V to the normal bundle of $B(G)$ in $T(G)$ which, by definition, is none other than $\omega(G)$. Thus the pair (V^{m+q}, M^m) defines a cobordism class of the right kind. Finally, we show that this cobordism class maps to the homotopy class of g. We have already said that g induces a bundle map $\xi \to \omega(G)$: if we use this map in the Thom construction, then the resulting $h : V^{m+q} \to T(G)$ agrees with g, together with its derivatives, on M^m. After a small homotopy, then, we can suppose $g = h$ on a neighbourhood of M. But the complement of such a neighbourhood is mapped, both by g and by h, to $T(G) \setminus B(G)$, which is contractible. It follows that $h \simeq g$, as desired.

We must now prove that the map is injective. But this follows by almost exactly the same arguments. Suppose given $M_0 \subset V \times 0$, $M_1 \subset V \times 1$ giving rise by the Thom construction to maps f_0, $f_1 \colon V \to T(G)$, and a homotopy $F \colon V \times I \to T(G)$ between f_0 and f_1. By (II, 5.1), we can replace F by a homotopy F' of f_0 to f_1, which is transverse to $B(G)$. Let $N = F'^{-1}(B(G))$. Then N is a submanifold of $V \times I$, and provides a cobordism of M_0 to M_1. Also, the normal bundle of N is induced from ωG, and so admits structure group G. Finally, this reduction to G induces the given reductions of the normal bundles of M_0, M_1 (since F' extends f_0 and f_1).

If V is a sphere S^{m+q}, (2.2) shows that we can use connected sum to define addition: we need not connect the submanifolds M as well, since we have not supposed them connected. Thus we use discs disjoint from the neighbourhood of M to define addition: these discs are mapped to ∞ by the Thom construction. If we then remove discs, and glue two spheres together, we obtain the usual sum of homotopy classes. □

This completes the proof of the theorem. Although the result is already extremely useful, we will go on to some important generalisations. However, these contain little extra in concept beyond the original result. The concept may perhaps bets be stated in terms of cobordism itself (we have already observed that homotopy is a special case of cobordism): it is that the extra structure defined by a submanifold whose normal bundle has group G is equivalent to the extra structure consisting of a map to $T(G)$ (at least, for cobordism theory).

Corollary 4.2. *Let G be a stable group. Then we have an isomorphism*

$$\Omega_n^G \cong \lim_{N \to \infty} \pi_{n+N}(T(G_N)).$$

Proof. By definition, possession of a G-structure is equivalent to having a normal G_N-structure in S^{N+n} for some N. If we fix N, then (by the theorem) we obtain the group $\pi_{n+N}(T(G_N))$. We claim that the desired group is the direct limit of these under the obvious injection maps: this again is essentially by definition. □

If G satisfies (**S**), then it is easily seen that $\pi_{n+N}(T(G_N))$ is independent of N for N large enough (we leave to the reader as an exercise to ascertain the precise value), so no limiting process is necessary.

A case if particular simplicity is $G = \{1\}$: each G_N consists only of the unit element. For each bundle occurring, then, an isomorphism with a trivial bundle is specified. Such an isomorphism we call a *framing* (it amounts to specifying a basis for each fibre $\mathbb{R}^N$), and we call the bundle *framed*. In this case, we take a point from $B(G_N)$; then $T(G_N) = S^N$, and

Corollary 4.3. *We have*

$$\Omega_n^{\{1\}} \cong \lim_{N \to \infty} \pi_{n+N}(T(G_N));$$

i.e., framed cobordism groups are isomorphic to stable homotopy groups of spheres.

This (due to Pontrjagin) was the first theorem in the subject.

We next discuss multiplicative structure. Let G, H be groups of orthogonal operators on $\mathbb{R}^q$, $\mathbb{R}^r$. Then $B(G) \times B(H)$ is a classifying space for $G \times H$, and $\omega(G) \times \omega(H)$ is a universal bundle. As to the Thom space (and this is a general remark for product bundles), the identifications to be made to $A(G \times H)$, which is homeomorphic to $A(G) \times A(H)$, to obtain $T(G \times H)$, include strictly these necessary to form $T(G) \times T(H)$: in fact, in this further space, we must identify $T(G) \times \infty \cup \infty \times T(H)$ to a point. If we use ∞ as base point in $T(G)$, this gives the "smash product", so we have

$$T(G \times H) = T(G) \wedge T(H).$$

However, we only need the existence of a map $T(G) \times T(H) \to T(G \times H)$ in order to define an external product

$$[V : T(G)] \times [W : T(H)] \to [V \times W : T(G) \times T(H)];$$

the induced map to $[V \wedge W : T(G) \wedge T(H)]$ is useful only in the case when V and W are spheres. This case provides

Corollary 4.4. *Suppose that G satisfies* (**M**)*, then products in Ω_*^G correspond to the pairing in homotopy groups induced by the maps*

$$T(G_M) \wedge T(G_N) \to T(G_{M+N}).$$

We now observe that these results can all be generalised to bordism groups.

Theorem 4.5. *If G is a stable group with countable homotopy groups, then Thom construction induces isomorphisms*

$$\Omega_M^G(X) \cong \lim_{N\to\infty} \pi_{M+N}((T(G_N)) \times X/(\infty \times X))$$

Proof. Let M^m be a submanifold of S^{m+N} whose normal bundle, ξ, has group G_N. Now we had a map $A_\xi \to A(G_N)$: we also have a projection $A_\xi \to M$. If we have a map $M \to X$, so that M defines a bordism class of X, we have a composite map $A_\xi \to M \to X$ and so, taking products, a map $A_\xi \to A(G_N) \times X$. This induces $\dot{A}_\xi \to \dot{A}(G_N) \times X$. Now shrink $\dot{A}_\xi$ to a point. We obtain maps

$$S^{m+N} \to T_\xi = A_\xi/\dot{A}_\xi \to (A(G_N) \times X)/(\dot{A}(G_N) \times X) = (T(G_N) \times X)/(\infty \times X).$$

Precisely as in Theorem 4.1, we see that this construction define a map

$$\Omega_m^G \to \pi_{m+N}((T(G_N \times X))/(\infty \times X)).$$

To check that the map is surjective, we start with

$$f : S^{m+N} \to (T(G_N \times X))/(\infty \times X),$$

and let K be the inverse image of $\infty \times X$. Then f defines a map of $S^{m+N} \setminus K$ to $(T(G_N) \setminus \{\infty\}) \times X$. We alter the first component on a compact subset of

$S^{m+N} \setminus K$ by a small homotopy, to make it transverse to $B(G_N)$. This defines also a homotopy of f, say to f'. Now set $M^m = f'^{-1}(B(G_N) \times X)$; then f' induces a map $M^m \to X$, and as before the normal bundle of M has group reduced to G_N. It follows, as before, that the bordism class defined by M maps to the homotopy class of f. Again, injectivity follows by a similar but simpler argument, and the proof that the bijection preserves group structures is the same as before. The passage to the limit works as before. □

Let us write X^O for the disjoint union of X and a point $*$, which we take as base point. Then

$$\begin{aligned} T(G_N) \wedge X^O &= (T(G_N) \times X \cup T(G_N) \times *)/(T(G_N) \times * \cup \infty \times X) \\ &= (T(G_N) \times X)/(\infty \times X). \end{aligned}$$

Thus the above result can be written more compactly as an isomorphism

$$\Omega_m^G(X) \cong \pi_{m+N}(T(G_N) \wedge X^O).$$

Note that $X^O \wedge Y^O = (X \times Y)^O$. You see, as in 4.3, that

Corollary 4.6. *Under the above isomorphism, external products*

$$\Omega_m^G(X) \times \Omega_n^G(Y) \to \Omega_{m+n}^G(X \times Y)$$

correspond to the homotopy pairings induced by

$$T(G_M) \wedge X^O \wedge T(G_N) \wedge Y^O \to T(G_{M+N}) \wedge (X \times Y)^O.$$

A similar argument to that of Theorem 4.5, but replacing S^{m+N} by a disc D^{m+N}, shows

Lemma 4.7. *With the assumption of (4.5), we have isomorphisms*

$$\Omega_m^G(X, Y) \cong \pi_{m+N}(T(G_N) \wedge X^O, T(G_N) \wedge Y^O)$$

for N large.

Chapter 5

Bordism as a homology theory.

We shall suppose throughout this chapter that G is a stable group. Then the inclusions $i_n\colon G_n \to G_{n+1}$ induces bundle maps $\omega(G_n) \oplus 1 \to \omega(G_{n+1})$, and hence maps of Thom spaces. Recalling that the Thom space of a Cartesian product is the smash product of the Thom spaces, we have

$$T_{\omega(G_n)\oplus 1} = T_{\omega(G_n)} \wedge S^1 = ST(G_n),$$

the suspension of $T(G_N)$. Thus we have maps

$$ST(G_n) \xrightarrow{i'_n} T(G_{n+1})$$

The sequence $\{T(G_n), i'_n\}$ is a spectrum: we will denote it by $\mathbb{T}(G)$. If G satisfies (**M**) and (**A**), the products $\psi_{m,n}\colon G_m \times G_n \to G_{m+n}$ similarly induce maps $\psi'_{m,n}\colon T(G_m) \wedge T(G_n) \to T(G_{m+n})$, and these associate up to homotopy. This provides $\mathbb{T}(G)$ with the structure of a ring spectrum.

Now any spectrum $\mathbb{A} = \{A_n, i_n\}$ gives rise to a homology theory on defining

$$\begin{aligned} H_n(X;\mathbb{A}) &= \lim_{N\to\infty} \pi_{n+N}(A_n \wedge X^O) \\ H_n(X,Y;\mathbb{A}) &= \lim_{N\to\infty} \pi_{n+N}(A_n \wedge X^O, A_n \wedge Y^O) \\ &= \lim_{N\to\infty} \pi_{n+N}(A_n \wedge X, A_n \wedge Y) \end{aligned}$$

and clearly if $\mathbb{A}$ is a ring spectrum we obtain associative external products. Hence the results of Chapter 4 can be summarised by

Theorem 5.1. *The Thom construction induces a natural equivalence between the functor Ω^G_* and homology theory with coefficients in the spectrum $\mathbb{T}(G)$; this respects products in the multiplicative case.*

It follows from this that Ω_*^G defines a homology theory; however, we prefer to present also a direct proof of this fact.

Theorem 5.2. *The groups $\Omega_*^G(X)$, $\Omega_*^G(X,Y)$ satisfy the axioms for a homology theory.*

Proof. We must first define the boundary homomorphism. If $f\colon (N,\partial M) \to (X,Y)$ gives a bordism class of (X,Y), then $f|\partial M$ gives a bordism class of Y. If $F\colon (W,\partial_c W) \to (X,Y)$ is a cobordism, then $F|\partial_c W$ is a cobordism between the boundary maps of $F|\partial_- W$ and $F|\partial_+ W$: thus restriction induces a map $\partial_m\colon \Omega_m^G(X,Y) \to \Omega_{m-1}^G(Y)$ which is compatible with disjoint union and hence a homomorphism.

Also, we have not yet made explicit the functorial dependence of $\Omega_m^G(X)$ on X. If $f\colon M \to X$ represents a class, and $\varphi\colon X \to Y$ is a map, then $\varphi\circ f\colon M \to Y$ determines a bordism class of Y. Again, it is clear that this construction defines a homomorphism $\varphi_*\colon \Omega_m^G(X) \to \Omega_m^G(Y)$. We can proceed similarly for pairs. The first two axioms (that Ω_m^G is a functor), and the third (that ∂_m is a natural transformation) are trivial. The fifth axiom states that $\varphi_0 \simeq \varphi_1\colon X \to Y$ implies $\varphi_{0*} = \varphi_{1*}$. Indeed if $f\colon M \to X$ represents an element of $\Omega_m^G(X)$, and $\Phi\colon X \times I \to Y$ is the given homotopy, then $\Phi_0(F \times 1_I)$ provides the required cobordism.

The fourth axiom states that if $i\colon Y \to X$ and $j\colon (X,\varnothing) \to (X,Y)$ are inclusions, the sequence

$$\cdots \longrightarrow \Omega_m^G(Y) \xrightarrow{i_*} \Omega_m^G(X) \xrightarrow{j_*} \Omega_m^G(X,Y) \xrightarrow{\partial} \Omega_{m-1}^G(Y) \longrightarrow \cdots$$

is exact: we next verify this. It is our first illustration of (2.3). Exactness at $\Omega_m^G(Y)$ is formal: a cobordism to the zero class in X can be identified with a representative of a class in $\Omega_m^G(X,Y)$, and vice-versa. Since ∂j_* takes a representative $g\colon M \to Y$ to the empty class, it is zero; conversely, if the class of $f\colon (N,\partial M) \to (X,Y)$ is annihilated by ∂, there is a G-manifold N with boundary ∂M such that $f|\partial M$ extends to a map $e\colon N \to Y$. Form M' by glueing N to M along ∂M; then e and f define $f'\colon M \to X$, representing a class in $\Omega_m^G(X)$. We say that the image of this under j_* is the class of (M,f). Indeed, $f' \times 1_I\colon M' \times I \to X$ provides the required cobordism, if we introduce a corner along $\partial M \times 0$, and agree that $\partial_-(M' \times I) = M \times 0$, $\partial_c(N \times I) = N \times 0$ and $\partial_+(M' \times I) = M' \times 1$. Similarly, if $g\colon M \to Y$ determines a class in $\Omega_m^G(Y)$, we can regard $g \times 1_I$ as a cobordism of jig to zero in $\Omega_m^G(X,Y)$. Finally, given an element of $\ker j_*$ and a cobordism W of the j_*-image of a representative to zero, we have $\partial_+ W = \varnothing$, $\angle_- W = \varnothing$, and $f\colon (W,\partial_c W) \to (X,Y)$. But we now reinterpret $W' = W$ but with $\partial_- W' = \partial_- W$, $\partial_+ W' = \partial_c W$: then W' is a cobordism of the given representative of $\ker j_*$ to $f\colon \partial_+ W' \to Y$, which is clearly in the image of i_*.

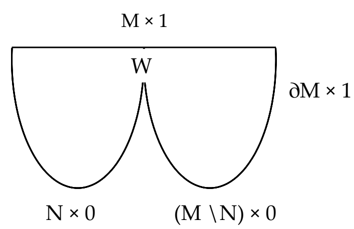
We must now check the excision axiom: that if $U \subset S$ has its closure in the interior of Y, then inclusion induces an isomorphism

$$\Omega_*^G(X \setminus U, Y \setminus U) \cong \Omega_*^G(X,Y).$$

To prove surjectivity, we let $f\colon (M,\partial M) \to (X,Y)$ represent an element of $\Omega_m^G(X,Y)$. It is convenient first to alter f (if necessary) by a homotopy on a collar neighbourhood of ∂M so that some smaller neighbourhood is mapped into Y. Then $A = f^{-1}(X \setminus Y)$ and $B = \partial M \cup f^{-1}(U)$ have disjoint closures, so we can find $s\colon M \to I$ with $s(A) = 0$ and $s(B) = 1$: in fact, since M is a compact metric space by (I, 1.2), we can set

$$s(P) = \rho(P,A)/(\rho(P,A) + \rho(P,B)).$$

We approximate s by a smooth map (as in 0, 2.2) and make it to 1/2 by (II, 5.1). Let $N = s^{-1}[0,1/2]$: then N is a smooth submanifold of M, and $f|N$ determines an element of $\Omega_m^G(X \setminus U, Y \setminus U)$. But N and M determine the same class in $\Omega_m^G(X,Y)$: for a cobordism W, we use $f \times 1_I\colon M \times I \to X$ with a corner introduces at $\partial N \times 0$ and the corner at $\partial M \times 0$ rounded (I, 6.10 and I, 6.5) - since $(M \setminus N) \subset s^{-1}[1/2,1]$, it is disjoint from A, and $f(M \setminus N) \subset Y$, so we can safely adjoin $(M \setminus N) \times 0$ to $\partial_c W$. The proof of injectivity is similar. If



$f\colon (W,\partial_c W) \to (X,Y)$ is a cobordism of $f|\partial_- W\colon (\partial_- W, \angle_- W) \to (X \setminus U, Y \setminus U)$ to $\partial_+ W = \varnothing$, we first adjust f so that $A = f^{-1}(X \setminus Y)$ and $B = \partial_c W \cup f^{-1}(U)$ have disjoint closures. Next choose a smooth $s\colon (W,A,B) \to (I,0,1)$, transverse to 1/2, and set $V = s^{-1}[0,1/2]$. Then V is a cobordism of $\partial_- V$ to zero in $\Omega_m^G(X \setminus U, Y \setminus U)$: a cobordism of $\partial_- V$ to $\partial_- W$ is obtained exactly as above. This completes the proof of the theorem. □

Various standard properties of homology now follows.

Corollary 5.3. *If (X,Y) is a CW pair, or more generally if it has the homotopy extension property, $\Omega_*^G(X,Y) \cong \Omega_*^G(X/Y, pt) \cong \tilde{\Omega}_*^G(X/Y)$.*

Proof. Under the assumption, X/Y has the homotopy type of X with a cone on Y attached; by excision, this modulo the cone has the same groups as X modulo Y. □

Corollary 5.4. *If X is the cone on Y, $\tilde{\Omega}_*^G(X) = 0$, and $\partial\colon \Omega_m^G(X,Y) \cong \tilde{\Omega}_{m-1}^G(Y)$.*

Proof. The first assertion follows from the homotopy axiom, the second from the exact sequence. □

Corollary 5.5. *If $X \supset Y \supset Z$ is a triple, we have an exact sequence*

$$\cdots \longrightarrow \Omega_m^G(Y,Z) \longrightarrow \Omega_m^G(X,Z) \longrightarrow \Omega_m^G(X,Y) \longrightarrow \Omega_{m-1}^G(Y,Z) \longrightarrow \cdots$$

Proof. This is a standard exercise in diagram chasing. □

Corollary 5.6. $\Omega_m^G(S^p) \cong \Omega_{m-p}^G$.

Proof. Follows by induction from the preceding two Corollaries. □

Definition 5.7. Let $X = Y_1 \cup Y_2$, $Z = Y_1 \cap Y_2$. We call $(X; Y_1, Y_2)$ a *proper triad* if inclusion induces isomorphisms

$$\Omega_*^G(Y_i, Z) \cong \Omega_*^G(X, Y_{1-i})$$

(By Corollary 5.3, this holds if all pairs (X, Y_i) and (Y_i, Z) have the homotopy extension property.)

Corollary 5.8. *If $(X; Y_1, Y_2)$ is a proper triad, we have the exact sequences*

$$\cdots \longrightarrow \Omega_m^G(Z) \longrightarrow \Omega_m^G(Y_1) \oplus \Omega_m^G(Y_2) \longrightarrow \Omega_m^G(X) \longrightarrow \Omega_{m-1}^G(Z) \longrightarrow \cdots$$

$$\cdots \longrightarrow \Omega_m^G(Z) \longrightarrow \Omega_m^G(X) \longrightarrow \Omega_m^G(X, Y_1) \oplus \Omega_m^G(X, Y_2) \longrightarrow \Omega_{m-1}^G(Z) \longrightarrow \cdots$$

Proof. These follow by another standard argument (the same for both). □

Corollary 5.9.

$$\Omega_*^G(X \cup Y) \cong \Omega_*^G(X) \oplus \Omega_*^G(Y) \quad \textit{for disjoint union.}$$
$$\tilde{\Omega}_*^G(X \vee Y) \cong \tilde{\Omega}_*^G(X) \oplus \tilde{\Omega}_*^G(Y) \quad \textit{if} \quad (X \vee Y; X, Y) \quad \textit{is proper.}$$

Proof. Apply the previous corollary. If $Z = \varnothing$, we certainly have a proper triad. □

Corollary 5.10. *If (X, Y) is a CW pair,*

$$\Omega_m^G(X^p \cup Y, X^{p-1} \cup Y) \cong C_p(X, Y; \Omega_{m-p}^G).$$

Proof. By (5.3),

$$\Omega_m^G(X^p \cup Y, X^{p-1} \cup Y) \cong \Omega_m^G(X^p/(X^{p-1} \cup (X^p \cap Y))).$$

But $X^p/(X^{p-1} \cup (X^p \cap Y))$ is a wedge of p-spheres; now apply (5.6) and (5.9). □

These corollaries all illustrate how we can begin to calculate the groups $\Omega_m^G(X, Y)$ in terms of the Ω_m^G (the calculation of these is postponed to Part VB[1]). After (5.10), we can formalise this process as a spectral sequence.

[1] Part VB has not been written.

Theorem 5.11. *Let (X, Y) be a CW pair. Then there is a first quadrant Ω_*^G-module spectral sequence, converging strongly to $\Omega_*^G(X, Y)$, which starts with $E_{pq}^2 = H_p(X, Y; \Omega_q^G)$.*

Proof. If $r < q < p$ we have, by (5.5), the exact bordism sequence of the triple $(X^p \cup Y, X^q \cup Y, X^r \cup Y)$: all the maps are induced by inclusions and boundary homomorphisms, so all expected diagrams commute. But such a collection of exact sequences always defines a spectral sequence. We write $X^\infty = X$, $X^{-\infty} = \varnothing$: then the end term is certainly $\Omega_*^G(X, Y)$. The module structure is induced by natural products $\Omega_m^G \times \Omega_n^G(X^p \cup Y, X^q \cup Y) \to \Omega_{m+n}^G(X^p \cup Y, X^q \cup Y)$: if M^m is a closed manifold, and $f\colon (N, \partial N) \to (X^p \cup Y, X^q \cup Y)$, then we use the manifold $M \times N$ (with induced G-structure) and the map induced by first projecting on N.

The E^1 term is imply

$$E_{pq}^1 = \Omega_{p+q}^G(X^p \cup Y, X^{p-1} \cup Y) \cong C_p(X, Y; \Omega_q^G)$$

by (5.10). The boundary d^1 is induced by taking the boundary of a manifold: we should next verify that this coincides with the usual boundary in the chain complex of (X, Y), as it then follows that $E_{pq}^2 = H_p(X, Y; \Omega_q^G)$ and hence that we have a first quadrant spectral sequence (evidently $\Omega_q^G = 0$ for $q < 0$). We omit the verification, which is a standard argument in homotopy theory.

As to convergence, we note that

$$\begin{aligned} \Omega_n^G(X^{-\infty} \cup Y) &= \Omega_n^G(X^p \cup Y) \quad \text{for all} \quad p < 0 \\ \Omega_n^G(X^p \cup Y) &= \Omega_n^G(X^\infty \cup Y) \quad \text{for all} \quad p > n, \end{aligned}$$

the first since $X^{-1} = \varnothing = X^\infty$ and the second since (by the cellular approximation theorem) any map of an n-manifold into X is homotopic to a map into X^n. These two isomorphisms imply strong convergence of the sequence. □

We shall defer explicit calculation till Part VB.[2] However, one useful reinterpretation may be noted here, which reduces yet further the problem of computing cobordism groups of pairs. Let G be as above, and H_q a topological group of orthogonal operators on $\mathbb{R}^q$. Then Lemma 3.5 produces the remark that setting $(G \times H_q)_n = G_{n-q} \times H_q$ defines a stable group $G \times H_q$, which satisfies (**S**) if G does.

Lemma 5.12. *We have $\Omega_n^{G \times H_q} \cong \Omega_{n+q}^G(T(H_q))$, and more generally*

$$\Omega_n^{G \times H_q}(X) \cong \Omega_{n+q}^G(T(H_q) \wedge X^0).$$

[2]Part VB has not been written.

Proof. By Theorem 4.5, we have

$$\begin{aligned}\Omega_n^{G\times H_q}(X) &= \lim_{N\to\infty} \pi_{n+N}(T(G\times H_q)_N \wedge X^0)\\ &= \lim_{N\to\infty} \pi_{n+N}(T(G_{N-q}\times H_q)\wedge X^0)\\ &= \lim_{N\to\infty} \pi_{n+N}(T(G_{N-q})\wedge T(H_q)\wedge X^0)\\ &= \Omega^G_{n+q}(T(H_q)\wedge X^0).\end{aligned}$$

□

Remark 5.13. Under favourable conditions, we also have a 'Thom isomorphism' of the last-mentioned group with $\Omega_n^G(B(H_q)\times X)$.

We have developed so far only homology theory associated with the spectrum $\mathbb{T}(G)$ and so with the stable group G. There is also an associated cohomology theory, defined by

$$\Omega^n_G(X) = H^n(X;\mathbb{T}(G)) = \lim_{N\to\infty}[S^N X : T(G_{N+n})].$$

Since we are not particularly concerned with general theory here, we only mention the geometric content of the above definition. This arises again by Theorem 4.5; this time we note that $S^N X$ is not a manifold, even if X is, but (if we take the reduced suspension) has only one 'bad point', whose complement is $\mathbb{R}^N\times X$. As we will always map the bad point to ∞, this does not matter. Then by (4.5), $[S^N X : T(G_{N+n})]$ corresponds bijectively to cobordism classes of submanifolds of $\mathbb{R}^N\times X$ whose normal bundles have group reduced to G_{N+n}.

Theorem 5.14. *Let G satisfy* (**M**), (**A**) *and* (**S**). *Let M^m have a weak G-structure. Then $\Omega^n_G(M)\cong\Omega^G_{m-n}(M,\partial M)$.*

Proof. In this case, $\mathbb{R}^N\times M^m$ also has a weak G-structure. By Lemma 3.1, a G_{N+n}-structure on the normal bundle of V^{m-n} in $\mathbb{R}^N\times M^m$ then induces a weak G-structure on the tangent bundle of V, and conversely if G is large enough. Combining this with the remark preceding the lemma, we have a bijective correspondence between $\Omega^n_G(M)$ and cobordism classes of manifolds V^{m-n} with weak G-structure and an imbedding in $\mathbb{R}^N\times M^m$, for large enough N. But if N is large, any map to $\mathbb{R}^N\times M^m$ is homotopic to an imbedding, and homotopic imbeddings are cobordant, by (II, 5.3). Hence specifying an imbedding in $\mathbb{R}^N\times M^m$ is equivalent to specifying a map to $\mathbb{R}^N\times M^m$ - or again, a map to M^m: it remains only to note that if M has boundary, ∂V is imbedded in $\mathbb{R}^N\times\partial M$, so we must insist that it be mapped to ∂M. □

Chapter 6

The classical exact sequences.

The sequences to which the title of this chapter refers were originally devised to relate $\Omega_*^{\mathbf{O}}$ and $\Omega_*^{\mathbf{SO}}$, as a means of calculating the latter. A more abstract proof was found by Atiyah (who invented bordism theory for the purpose), and we present a generalisation of an improvement due to Conner and Floyd, who considered the case of $\Omega_*^{\mathbf{U}}$ and $\Omega_*^{\mathbf{SU}}$. We will then give the geometrical proofs too.

Let G be a stable group, defined by a sequence

$$\cdots \longrightarrow G_{n-1} \xrightarrow{i_{n-1}} G_n \xrightarrow{i_n} G_{n+1} \longrightarrow \cdots$$

where G_n operates on $\mathbb{R}^n$. Let $SG_n \subset G_n$ be a sequence of normal subgroups, with $i_n(SG_n) \subset G_{n+1}$, and that i_n induces isomorphisms of G_n/SG_n. This last condition could perhaps be weakened to requiring that each homotopy group $\pi_r(G_n/SG_n)$ becomes independent of n, for large n. We will denote by Z the quotient group $\lim_{n\to\infty} G_n/SG_n = G/SG$, say.

We will also suppose that G satisfies (**M**), and that the subgroups SG_n are stable under the product maps ψ.

The examples w have particularly in mind are when $Z = \mathbf{O}_1 (\cong \mathbb{Z}_2)$ and $G = \mathbf{O}$ or **Pin**, $SG = \mathbf{SO}$ resp. **Spin** or when $Z = \mathbf{U}_1 (\cong S^1)$ and $G = \mathbf{U}$ or $\mathbf{Spin}^c$, $SG = \mathbf{SU}$ resp. **Spin**. The following is also a useful construction. Let H be any topological group. Then we can replace G by $G \times H$ and SG by $SG \times H$, where $G_n \times H$ operates on $\mathbb{R}^n$ via its projection on G_n. Note that $B(G_n \times H) = B(G_n) \times B(H)$, $T(G_n \times H) = T(G_n) \wedge B(H)$. In particular, if X is any CW complex, the loop space ΩX is equivalent to a topological group, and we have

$$\begin{aligned}\Omega_n^{G\times\Omega X} &= \lim_{N\to\infty} \pi_{n+N}(T(G_{n+N} \times \Omega X)) \\ &= \lim_{N\to\infty} \pi_{n+N}(T(G_{n+N} \wedge X)) = \tilde{\Omega}_n^G(X).\end{aligned}$$

This allows us to consider only coefficient groups of homology theories, and later to deduce their general values.

Theorem 6.1. *Let G, SG and Z be as above. Let α be a G_k-bundle over BZ whose classifying map induces, via $BZ \to BG_k \to BG \to B(G/SG) \to BZ$, a homotopy equivalence. Then $\Omega_n^G \cong \tilde{\Omega}_{n+k}^{SG}(T(\alpha))$.*

Proof. Let χ be the classifying map of α. Denote by f_N the composite of

$$B(SG_N) \times BZ \xrightarrow{i\times\chi} B(G_N) \times B(G_k) \xrightarrow{B_{\psi_{N,k}}} B(G_{N+k})$$

These maps are compatible with i_N, hence there is a limit map $f\colon B(SG) \times BZ \to BG)$. We claim that f induces isomorphism of homotopy groups, this is clear from the definition of f, the exact sequence

$$\cdots \to \pi_r(SG) \to \pi_r(G) \to \pi_r Z \to \pi_{r-1}(SG) \to \cdots,$$

and the fact that (up to automorphism) χ_* splits the projection $\pi_r(G) \to \pi_r Z$.

Now by definition, f_N is covered by a bundle map of the direct sum of the universal bundle over $B(SG_N)$ and α to the universal bundle over $B(G_{N+k})$. Thus we also have a map

$$g_N\colon T(SG_N) \wedge T(\alpha) \to T(G_{N+k}).$$

Since f_N induces homotopy isomorphisms in the limit, so does g_N.

We now have

$$\begin{aligned}\Omega_n^G &= \lim_{N\to\infty} \pi_{N+n+k}(T(G_{N+k}))\\ &= \lim_{N\to\infty} \pi_{N+n+k}(T(SG_N) \wedge T(\alpha))\\ &= \bar{\Omega}_{n+k}^{SG}(T(\alpha)).\end{aligned}$$

□

The next result is a companion to (6.1), but needs less hypotheses. It is related to the Thom isomorphism theorem.

Theorem 6.2. *Let G be a stable group satisfying* (**M**), *P a topological space, α a G_k-bundle over P. Then $\Omega_n^G(P) \cong \tilde{\Omega}_{n+k}^G(T(\alpha))$.*

Proof. Let χ classify α, f_N denote the composite

$$B(G_N) \times P \xrightarrow{1\times\chi} B(G_N) \times B(G_k) \xrightarrow{B_{\psi_{N,k}}} B(G_{N+k}),$$

and F_N the map $B(G_N) \times P \to B(G_{N+k}) \times P$ whose components are f_N and projection on the second factor. F_N is covered by a bundle map of the direct sum of ω_N and α to ω_{N+k}. Also, $B(G_N)$ is mapped by the natural injection i to $B(G_{N+k})$, and we have a commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_r(B(G_N)) & \longrightarrow & \pi_r(B(G_N)\times P) & \longrightarrow & \pi_r(P) & \longrightarrow & 0\\ & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle F_{N*}} & & \| & & \\ 0 & \longrightarrow & \pi_r(B(G_{N+k})) & \longrightarrow & \pi_r(B(G_{N+k})\times P) & \longrightarrow & \pi_r(P) & \longrightarrow & 0.\end{array}$$

Thus F_{N*} is an isomorphism in the limit as $N \to \infty$. We have an induced map of Thom spaces

$$T(G_N) \wedge T(\alpha) \to T(G_{N+k}) \wedge P^O,$$

which then also in the limit gives homotopy isomorphisms. The conclusion of the proof is now as before. □

Corollary 6.3. *With the hypotheses of 6.1, if β is an SG_k-bundle over BZ, we have an isomorphism*

$$\Omega_n^{SG}(BZ) \cong \tilde{\Omega}_{n+k}^{SG}(T(\beta)).$$

To obtain exact sequences from these results we need some restriction on BZ - or rather, on Z. We will now assume that either $Z = \mathbf{O}_1 \cong \mathbb{Z}_2$ or $Z = \mathbf{U}_1 \cong S^1$. Correspondingly, $BZ = P$ (say) is infinite real, resp. complex, projective space. Let us write $d = 1$ in the first case and $d = 2$ in the second.

The following will be useful for checking the hypothesis of (6.1). Since BZ is an Eilenberg-MacLane space, a map $BZ \to BZ$ is a homotopy equivalence if and only if it induces an automorphism of the homotopy group - or equivalently, of the lowest homology group.

We will make the further assumption that the standard real of complex line bundle η over P is a G_d-bundle, inducing a homotopy equivalence (which must be, up to sign, the identity) $P \to P$. This is easily verified in each of the cases mentioned earlier. In the complex case, the conjugate $\bar{\eta}$ is then also a G_2-bundle. We now take $\alpha = (m+1)\eta + m\bar{\eta}$. Since the first Stiefel-Whitney (resp. Chern) class of this is a generator, we can apply Theorem 6.1. To compute P^α, we note that if the structure group is extended to $\mathbf{O}_{2m+1}$, α becomes equivalent to $(2m+1)\eta$, and so P^α is homeomorphic to P/P_{2m}, where P_{2m} is the sub-projective space of dimension $2m$. This proves

Corollary 6.4. *With the above assumptions,*

$$\Omega_n^G \cong \tilde{\Omega}_{n+(2m+1)d}^{SG}(P/P_{2m}).$$

We also apply 6.3 with $\beta = 2m\eta$, so $P^\beta = P/P_{2m-1}$, to obtain

Corollary 6.5. $\Omega_n^{SG} \cong \tilde{\Omega}_{n+2md}^{SG}(P/P_{2m-1})$.

Also note that $\Omega_n^{SG}(P) \cong \tilde{\Omega}_n^{SG}(P) \oplus \Omega_n^{SG}$, and that taking $m = 0$ in 6.4, $\Omega_n^G(P) \cong \tilde{\Omega}_n^{SG}(P)$. Putting these together, we have

Corollary 6.6. $\Omega_{n-d}^G \oplus \Omega_n^{SG} \cong \tilde{\Omega}_{n+2md}^{SG}(P/P_{2m-1})$.

We now obtain the exact sequences.

Theorem 6.7. *Let G be a stable group satisfying* (**M**), *SG a subgroup with $G/SG_n = \mathbf{O}_1$ or $\mathbf{U}_1$ for $n \geqq d$ ($d = 1$ for $\mathbf{O}_1$, 2 for $\mathbf{U}_1$), and stable for ψ, $P = P_\infty(\mathbb{R})$ or $P_\infty(\mathbb{C})$. Suppose the standard line bundle η over P is a G_d-bundle, inducing a map $P \to P$ homotopic to the identity. Then there are exact sequences (where $P_2 = P_2(\mathbb{R})$ or $P_2(\mathbb{C})$).*

(i)

$$\cdots \longrightarrow \Omega_n^{SG} \longrightarrow \Omega_n^G \longrightarrow \Omega_{n-d}^{SG} \oplus \Omega_{n-2d}^{SG} \longrightarrow \Omega_{n-1}^{SG} \longrightarrow \cdots$$

(ii)

$$\cdots \longrightarrow \tilde{\Omega}_{n+d}^{SG}(P_2) \longrightarrow \Omega_n^G \longrightarrow \Omega_{n-2d}^{SG} \longrightarrow \Omega_{n+d-1}^{SG}(P_2) \longrightarrow \cdots$$

(iii)

$$\cdots \longrightarrow \Omega_n^{SG} \longrightarrow \tilde{\Omega}_{n+d}^{SG}(P_2) \longrightarrow \Omega_{n-d}^{SG} \longrightarrow \Omega_{n-1}^{SG} \longrightarrow \cdots$$

Proof. (i) $P_1 \subset P$ is a sphere S^d. The sequence of spaces

$$S^d \to P \to P/P_1$$

has an exact homology sequence for Ω_*^{SG}. Also, we have $\tilde{\Omega}_{n+d}^{SG}(S^d) \cong \Omega_n^{SG}$ by (5.6), $\tilde{\Omega}_{n+d}^{SG}(S^d) \cong \Omega_n^G$ by (6.4) with $m = 0$, and $\tilde{\Omega}_{n+d}^{SG}(P/P_1) \cong \Omega_{n-2d}^{SG} \oplus \Omega_{n-d}^{SG}$ by (6.6) with $m = 1$. This gives (i).
(ii) Replace P_1 by P_2 in the above, and use the fact ((6.4) with $m = 1$) that $\tilde{\Omega}_{n+d}^{SG}(P/P_2) \cong \Omega_{n-2d}^G$.
(iii) Here we note that P_2/P_1 is a sphere S^{2d}, and use the exact $\tilde{\Omega}_*^{SG}$-sequence of $S^d \to P_2 \to S^{2d}$. □

We now turn to the geometrical approach, and will give a second complete proof, at the same time giving a more precise description of the maps in the sequences. Our second proof will illustrate sequence (i) as of the type described in (2.3); we will give a full discussion of this, and the rest will then follow. We will also improve several details of the theorem.

Now (2.3) gives us an exact sequence in which the third term is the cobordism group $\Omega_m^{G,SG}$ of bounded G-manifolds with an SG-structure on the boundary. We will evaluate this using the idea introduced after (2.3).

Let us agree, in order to avoid unnecessarily complicated notation below, that the G-structure of a manifold M is specified by the classifying map of its stable normal bundle, $\nu_m \colon M \to BG$: that we have a fibration

$$BSG \longrightarrow BG \xrightarrow{\pi} P;$$

and that an SG-structure of M is determined by a null-homotopy of $\pi \circ \nu_m$ which is thus covered by a homotopy of ν_m to a map into BSG. We shall also need the G-structure on the standard line bundle over P, classified by

$$P \xrightarrow{\eta} BG_d \xrightarrow{\iota} BG;$$

here we may assume that $\pi \circ \iota \circ \eta$ is the identity map of P, 1_P. We write $(-1)_P$ for the negative of the identity; in the real case, we can take $(-1)_P = 1_P$, and in the complex case, define $(-1)_P$ by complex conjugation. Now P is an H-space, and the diagram

$$\begin{array}{ccc} BG \times BG & \xrightarrow{B_\psi} & BG \\ \downarrow{\scriptstyle \pi\times\pi} & & \downarrow{\scriptstyle \pi} \\ P \times P & \longrightarrow & P \end{array}$$

is homotopy commutative; we shall alter (if necessary) out model of BG to make it commutative.

Now if M^m is a G-manifold, we consider the map $\pi \circ \nu_m \colon M \to P$. Altering by a homotopy, if necessary, we may suppose that this maps M to a finite dimensional projective subspace P_k. By (II, 4.6), we can make this map transverse to the submanifold P_{k-1}, whose preimage will then be a smooth submanifold V^{m-d} of M^m, with normal bundle induced from η. Moreover, if ∂M has an SG-structure, $\pi \circ \nu_m$ is trivial on ∂M (which has trivial normal bundle in M), so may be assumed to avoid P_{k-1}. Thus V lies in the interior of M, and is closed.

We now give V^{m-d} an SG-structure. Indeed, the stable normal bundle of V is the sum of the bundles induced from ν_m and from η; i.e., is induced by

$$V \subset M \xrightarrow{\nu_m} BG \xrightarrow{1\times\pi} BG \times P.$$

We shall give the second summand *minus* the obvious structure. So the normal bundle ν_V is now induced by

$$V \xrightarrow{\nu_m|V} BG \xrightarrow{1\times\pi} BG \times P \xrightarrow{1\times -1} BG \times P \xrightarrow{1\times\eta} BG \times BG \xrightarrow{B_\psi} BG.$$

The composite $\pi \circ \nu_V$ is thus induced by

$$V \xrightarrow{\nu_m|V} BG \xrightarrow{\pi} P \times P \xrightarrow{(1,-1)} P \times P \longrightarrow P,$$

and if we fix (once for all) a null-homotopy of the composite map $P \to P$, we define one for $\pi \circ \nu_V$, and hence an SG-structure for V.

Now we showed in Chapter 2 that M was (G, SG)-cobordant to a tubular neighbourhood of V. This is a bundle over V, with fibre D^d, associated to $(\pi \circ \nu|V)^*\eta$; hence its (G, SG)-cobordism class is determined by the class of $(V, \pi \circ \nu|V)$ in $\Omega^{SG}_{m-d}(P)$. The formula which determines it is as follows. Let η' be the bundle induced from η. Then $\nu_V = \nu_m + \bar{\eta}'$, where the bar recalls the sign change above. Thus $\nu_V + \eta' = \nu_m + \bar{\eta}' + \eta' = \nu_m + 2\varepsilon$ (ε a trivial G_d-bundle). It is clear from this that given any element of $\Omega^{SG}_{m-d}(P)$, represented say by (V, f), we can take the bundle E with fibre D^d associated to $f^*\eta$ and give it a G-structure.

Moreover, the stable normal bundle $\nu\partial E$ of the boundary ∂E is the restriction of ν_E. But $\pi \circ \nu_E$ is essentially f, by definition, and is covered by a bundle map over V of E to the disc bundle associated to η, and hence of ∂E to the corresponding sphere bundle Σ. But Σ is contractible, so we have a well defined null-homotopy of $\partial E \to \Sigma \to P$, and so an SG-structure on ∂E. Since all our constructions can - as in chapter 4 - be carried over for cobordism, we have an isomorphism $\Omega_m^{G,SG} \cong \Omega_{m-d}^{SG}(P)$.

We now wish to use the remark immediately preceding (4.2) that the extra structure provided by a submanifold gives the same cobordism group as the extra structure provided by a map to its Thom space; and combine this with the remark that P is homeomorphic to the Thom space of η. The details resemble those above: we have a map (χ_V say), of V to P, or more precisely to P_{k-1}. We make this transverse to P_{k-2}, and write $B = \chi_V^{-1}(P_{k-2})$. Then

$$\nu_B = \nu_V|B + (\chi_V|B)^*\eta,$$

and we use this formula to give B a G-structure. Our construction again works for cobordisms; since the class of (V, f) determines the cobordism classes of V and B, we have a homomorphism

$$\Omega_{m-d}^{SG}(O) \to \Omega_{m-d}^{SG} \oplus \Omega_{m-2d}^{G}.$$

In fact this is an isomorphism, for the class of (V, f) is determined by that of (V, B, and the map $B \to P$ inducing the normal bundle of B in V); by Corollary (3.6) we can separate the two elements of the pair, provided the stable normal bundle of B is induced by $B \to B(SG) \times P$ and finally, by the proof of (6.1), this latter is homotopy equivalent to $B(G)$.

We have thus obtained sequence (3.2); to complete the discussion, we must determine the boundary map

$$\Omega_{m-d}^{SG} \oplus \Omega_{m-2d}^{G} \to \Omega_{m-1}^{SG}.$$

As to the first component, we can suppose B empty and χ_V trivial. Then the disc bundle is trivial, and has boundary $V \times S^{d-1}$. This describes it as a G-manifold; for the SG-structure we must be more careful. All the construction is that of a product, hence we obtain multiplication by the class, α say, of S^{d-1} with appropriate SG-structure. To determine this, we can take V to be a point and M a disc D^d. Recall that V was constructed from M by making $\pi \circ \nu_m \colon M \to P_k$ transverse to P_{k-1}. Now $\partial M = S^{d-1}$ was mapped to a point by this, so $S^d = M/\partial M$ is mapped to meet P_{k-1} transversely in just one point. This coincides (up to homotopy) with the inclusion of a projective line P_1. So α is the class of S^{d-1}, with SG-structure defined by a framing of the normal bundle, twisted in this way. One can analyse the twisting more in general, but it is by now easier to remark that when $d = 1$ we have S^0, and each point has the positive orientation (this twists the standard framing of ∂D^1 by changing a sign). Thus in this case the map $\Omega_{m-1}^{SG} \to \Omega_{m-1}^{SG}$ is just multiplication by 2. In the case $d = 2$ we have S^1, and the twisted framing differs from the standard one. Here elementary homotopy theory tells us that $2\alpha = 0$.

Write (d_1, d_2) for the components of the map $\Omega_m^G \to \Omega_{m-d}^{SG} \oplus \Omega_{m-2d}^{G}$, so that the image of the class of M by d_1 (resp. d_2) is determined by V (resp. B). We now construct a map $\varphi\colon \Omega^G_{m-2d} \to \Omega^G_m$ and show that $d_1 \circ \varphi = 0$ and $d_2 \circ \varphi = id$. From this, and the exactness of the sequence

$$\Omega^G_m \xrightarrow{(d_1,d_2)} \Omega^{SG}_{m-d} \oplus \Omega^G_{m-2d} \xrightarrow{\binom{\times\alpha}{c}} \Omega^{SG}_{m-1}$$

now follows the second component (c) of the boundary map vanishes.

Suppose then that B^{m-2d} is a G-manifold, form $(\pi \circ \nu_B)$, which we may take as a map $B \to P_k$ for appropriate k. Then $\eta + \varepsilon^{2d}$ can be regarded as a real (esp. complex if $d = 2$) bundle over P_k; we form the associated projective bundle Q_{k+2}, and let M^m be the induced bundle over B, V^{m-d} the subbundle corresponding to $\eta+\varepsilon^d$, and identify B itself with the subbundle of V corresponding to η. It is well known that if to τ_M we add the bundle induced by $M \to B \to P_k$ from η, the result is the sum of a bundle induced by τ_B and three (real or complex) line bundles, corresponding to η, ε^d, and ε^d; and all induced from η, say by maps f_1, f_2 and f_2. We give these the F-structures induced by f_1, f_2 and $-1 \circ f_2$: this defines a G-structure on M, and as the construction applies to cobordisms and to disjoint unions, we have defined the desired map φ.

Although the G-structure itself is somewhat complicated, it is easy to see that $\pi \circ \nu_M$ is induced via the bundle map $\beta\colon M \to Q_{k+2}$ covering the original map $B \to P_k$. We will now write down a map $\zeta\colon Q_{k+2} \to P_{k+2}$ which is transverse to P_{k+1} and P_k, which have preimages the sub-bundles associated to $\eta + \varepsilon^d$ and η. Since Q is explicit, it is easy to see that $\zeta \circ \beta \simeq \pi \circ \nu_M$: thus M gives rise to V and B in the usual way. Hence we have $d_2 \circ \varphi = id$. To see $d_1 \circ \varphi = 0$, we must find an SG-manifold with boundary V: in fact, as V is a $P_1(= S^d)$-bundle over B, we take the associated disc bundle: such exists since the group of the bundle is not the full projective group but only $Z(= S^{d-1})$, and is topologically the product by I of the mapping cylinder of the principal bundle. Since the principal bundle was obtained from $\pi \circ \nu_B$, this has an SG-structure.

It remains to construct ζ: for this we follow Atiyah. Let $k = \mathbb{R}$ or $\mathbb{C}$ in the cases $Z = S^0$ or $S1$, and consider decomposition $k^{K+3} = k^{K+1} \oplus k^2$. Then P_K is the projective space of k^{K+1}, and we can identify the fibre of $\eta + \varepsilon^{2d}$ over the line $l \subset P_K$ with $l \oplus k^2$, and so Q_{K+2} with the subspace of $P_K \times P_{K+2}$ of pairs (l, m) of lines with $m \subset l \oplus k^2$. We take ζ as projection on P_{K+2}; so for $\zeta^{-1}(P_{K+1})$ we need $m \subset l \oplus k \oplus 0$ and for $\zeta^{-1}P_K$, $m = l$. Both transversalities are clear. We have established

Theorem 6.8. *There is an exact sequence*

$$\Omega^{SG}_n \xrightarrow{r} \Omega^G_n \xrightarrow{(d_1,d_2)} \Omega^{SG}_{n-d} \oplus \Omega^G_{n-2d} \xrightarrow{\binom{\times\alpha}{0}} \Omega^{SG}_{n-1} \longrightarrow \cdots$$

where α is the class of S^{d-1} with a twisted framing. Also, there exists $\varphi\colon \Omega^G_{n-2d} \to \Omega^G_n$ with $(d_1, d_2) \circ \varphi = (0, 1)$.

Corollary 6.9. *Write* $\Omega_n^{RG} = \ker d_2$*. Then there is a split exact sequence*

$$0 \longrightarrow \Omega_n^{RG} \xrightarrow{\ i\ } \Omega_n^G \underset{\varphi}{\overset{d_2}{\rightleftarrows}} \Omega_{n-2d}^G \longrightarrow 0.$$

Moreover, the following sequence is exact:

$$\Omega_n^{SG} \xrightarrow{\ s\ } \Omega_n^{RG} \xrightarrow{\ d_1\ } \Omega_{n-d}^{SG} \xrightarrow{\ \times\alpha\ } \Omega_{n-1}^{SG} \longrightarrow \cdots$$

Here, r is the forgetful map and i the inclusion; the first sequence shows that r factorises as $r = is$, and the corollary is immediate. Moreover, on comparing the above with Theorem 6.7, we are led to the identification

$$\Omega_n^{RG} = \bar{\Omega}_{n+d}^{SG}(P_2).$$

In fact yet another definition is sometimes more convenient: Ω_m^{RG} is the cobordism group of G-manifolds M^m provided with a homotopy of $\pi \circ \nu_M : M \to P$ to a map into $P_1 (= S^d)$. For the corresponding V is then mapped to P_0 and B to P_{-1}, so B is empty, so such manifolds lie in $\ker d_2$. Conversely, if M is in $\ker d_2$, an extension of cobordisms argument shows that B may be supposed empty. But then the image of $\pi \circ \nu_M : M \to P_k$ avoids P_{k-2}, so is homotopic (by an obvious projection) to a map to the complementary P_1.

Chapter 7

Equivariant Cobordism

The object of this chapter is to give a programme for reducing the calculation of equivariant cobordism groups to that of the bordism groups of certain classifying spaces. It will first be necessary to develop thoroughly the foundations of the theory of smooth group actions.

Let H be a compact Lie group, M a smooth manifold (perhaps with boundary, or corner) and let

$$\varphi\colon M \times H \to M$$

define a smooth action of H on M. For each $P \in M$, write

$$H_P = \{h \in H \colon \varphi(P, h) = P\}.$$

Then H_P is a closed subgroup, called the *isotropy group* of P. We have

$$\varphi(P, h_1) = \varphi(P, h_2) \Leftrightarrow \varphi(P, h_1 h_2^{-1}) = P \Leftrightarrow h_1 h_2^{-1} \in H_P \Leftrightarrow H_P h_1 = H_P h_2.$$

It follows that φ induces a bijection ψ of the space of right cosets H/H_P onto the set of points $\varphi(P, h)(h \in H)$ - which is called the *orbit* of P. It also follows that $H_{\varphi(P,h)} = h^{-1} H_P h$. Thus the isotropy groups at the point of an orbit form a complete conjugate set of closed subgroups of H. Such sets are called *orbit types*, and the set containing H_P is the type of the orbit of P.

Lemma 7.1. *The orbit of P is a smooth submanifold of M, and ψ is a diffeomorphism.*

Proof. (1) Since H/H_P is compact and ψ injective, we know that ψ is a topological imbedding in M.

(2) Since φ is a smooth map, so is ψ

(3) It is now sufficient to show that $d\psi$ is everywhere injective.

(4) Now ψ is an equivariant map for smooth H-actions: translating by elements of H, we see that if $d\psi$ is injective at the unit element, it is injective everywhere, and conversely.

Suppose then $d\psi$ not injective anywhere. By a result of A. Sard 'Image of critical sets' Ann. of Math. 68 (1958) 247-259, if r is the topological dimension of H/H_P, the Hausdorff r-dimensional measure of $\psi(H/H_P)$, the orbit of P, is zero. By Theorem VII. 3 of W. Hurewicz and A. Wallman, 'Dimension theory', the dimension of $\psi(H/H_P)$ is $\leqq r-1$. This contradicts the fact that ψ is an imbedding, and proves the lemma. □

Now let V be the set of points of M with isotropy group H_P, a the set of conjugates of H_P, and W_a the union of the orbits with type a, so that we have $W_a = \varphi(V \times H)$. Let N_P be the normaliser of H_P in H: then V is invariant under the induced action of N_P.

Theorem 7.2. *V and W_a are smooth submanifolds of M and φ induces a diffeomorphism of $V \times_{N_P} H$ onto W_a.*

We do not assert that all components of V (or of W_a) have the same dimension.

Proof. We first assert that M admits a Riemannian metric which is invariant under the action of H. Indeed, by we have a metric μ; now the action of H on M induces an action on the Riemann bundle, and we will use $\nu = \int_H \mu^h dh$, where integration is with respect to Haar measure on the compact group H. Since positive definite symmetric matrices form a convex set, we obtain a positive definite scalar product on each tangent space, and the cross-section ν is clearly smooth.

Now consider the exponential map $\exp\colon M_P \to M$. Since we have an H-invariant metric, and $H_P \subset H$ operates in an induced way on M_P, exp is equivariant for the actions for H_P. In particular, the action of $h \in H_P$ on M is determined locally at P by the action on M_P which is linear - and, indeed, orthogonal. So the action of H_P on M near P is locally isomorphic to the action on Euclidean space given by an orthogonal representation φ of H_P. In particular, the set V_1 of fixed points of any subgroup H_1 of H_P corresponds to a linear subspace of M_P, and hence is a smooth submanifold.

Write O_P for the tangent space at P to the orbit of P; let S_P be its orthogonal complement, S'_P a small enough ε-neighbourhood of 0 in S'_P, and $S = \exp S'_P$. Any element of H_P leaves O_P invariant (it is invariantly defined), hence also S_P, S'_P, and S. Now since by (7.1) $d\psi$ is onto Q_P, it follows that orbits of S fill up a neighbourhood of P. Also φ induces a map

$$\chi\colon S \times_{H_P} H \to M$$

which, by the above, is a smooth immersion. Since the orbit of P is imbedded, so (by 0, 2.18) is some neighbourhood of it. Thus if ε is small enough, χ is an imbedding.

We deduce first that the orbit types of all points near P - which are the types of orbits of points Q of S - have $H_Q \subset H_P$: they are the isotropy groups of the action of H_P on S. Since $\dim S < \dim M$ we deduce by induction on $\dim M$ that there are only a finite number of orbit types near P, and hence that the set of points with isotropy group H_1 is an open subset of the set fixed by H_1. So V is smooth. It is immediate that φ induces a bijection of $V \times_{N_P} H$ onto W; it follows from Lemma 7.1 that we have a diffeomorphism. □

Now we have laid the foundations of the theory of smooth actions of compact groups, we can return to our cobordism problem. Observe that any point of the closure of V is fixed under H_P. Thus to ensure that V is a closed submanifold (or equivalently, that W is), it is sufficient to require that a is maximal in the orbit types of the given action.

The following special case is easily solved, and will be a pattern for the general result 7.5. Let $A = \{1\}$ contain the unit subgroup only. Then the action of H^h on M^m must be free. Thus M has the structure of a principal bundle with group and fibre H, and base X^{m-h}, say (the orbit space of the action): by the results of 7.1 and 7.2, X is also a smooth manifold. Let $\chi\colon X \to BH$ classify the bundle. Then the bordism class of χ belongs to $\Omega^{\mathbf{O}}_{m-h}(BH)$.

Lemma 7.3.

$$I^{\mathbf{O}}_m(H; \{1\}) \cong \Omega^{\mathbf{O}}_{m-h}(BH).$$

Proof. If W is a cobordism on which H acts freely, the orbit space W/H is a cobordism, mapping into BH: thus the two ends of W determine the same bordism class in BH, and we have a well-defined map $I^{\mathbf{O}}_m(H; \{1\}) \to \Omega^{\mathbf{O}}_{m-h}(BH)$.

To see that the map is surjective, note that BH can be replaced by a smooth manifold (see Chapter 4 above) and f by a smooth map, so we need only consider smooth bundles. Now given $f\colon X \to BH$, we consider the induced principal bundle over X with group H: this is a smooth m-manifold on which H operates freely, so defines an element of $I^{\mathbf{O}}_m(H; \{1\})$ which maps to the bordism class of f. Note that BH can be replaced by a smooth manifold (see Chapter 4) and f by a smooth map, we need only consider smooth bundles. Similarly it is injective, for if M and M' are such that M/H, M'/H define the same bordism class, we let $g\colon W \to BH$ denote a cobordism, and note that the induced principal H-bundle over W gives the required cobordism of M to M'. □

We continue our investigation of W_a: our main aim is the exact sequence of (7.4). We will suppose that the orbit type a is maximal for the given action (i.e., that if $H_P \in a$, H_P is not strictly contained in any H_Q). Let N_a be an ε-neighbourhood of W_a in the invariant metric. Then the usual projection (I, 2.14) which gives N the structure of disc bundle over W_a is an equivariant map. We are thus let to consider the following objects:

$\pi\colon N \to W_a$ is the projection of a smooth disc bundle; we identify W_a with the zero cross-section. The group H acts on N_a and W_a; π is equivariant, and the orbit type of a point of W_a is a; at other points of N_a, the orbit type is different (hence is less than a). We have $\dim N_a = m$; the components of W_a may have different dimensions.

For our exact sequence we incorporate one further element of structure. Let G be a stable group satisfying (**M**), (**A**) and (**S**), and M have a G-structure (on its stable tangent bundle). Suppose the compact Lie group H operates smoothly on M. We will say that H *respects the G-structure* if the following condition is satisfied. For some n, we are given an action of H on the principal G_n-bundle P which defines the G-structure, lifting the given action of H on M. This defines actions of H on the associated bundles; in particular, on the principal G_{n+1}-bundle, so the condition is independent of n.

Write $I_m^G(H;A)$ for the group of cobordism classes of manifolds M^m with G-structure and an H-action which respects it, and such that each orbit type belongs to the set A. We choose a maximal element a of A, and write $A' = A \setminus \{a\}$.

Write $A_m^G(H;A',a)$ for the group of cobordism classes of manifolds W with a smooth disc bundle $\pi\colon N^m \to W$ such that N^m is as above, π is equivariant (where W is identified with the zero cross-section), and the orbit type at a point of W is a; at other points of N belongs to A'.

The following illustrates (2.3) and the remark following it.

Theorem 7.4. *There is an exact sequence*

$$I_m^G(H;A') \xrightarrow{\ \alpha\ } I_m^G(H;A) \xrightarrow{\ \beta\ } A_m^G(H;A',a) \xrightarrow{\ \gamma\ } I_{m-1}^G(H;A') \xrightarrow{\ \alpha\ } I_{m-1}^G(H;A).$$

Proof. First we define the maps. Set α the natural map induced by taking the same representative. Next, if M admits an action with orbit types $\in A$, form W_a and N_a as above to define β. As to γ, take the class of the boundary ∂M.
(α,β) *is exact* $\beta\alpha = 0$, for if the orbit types of M belongs to A', we have $W_a = \varnothing$. Conversely, let W_a bound X and L be the corresponding disc bundle over X, so that $\partial_c L = N_a$ and $\partial_+ L$ is the sphere bundle over X. Attach L to $M \times I$ by glueing $\partial_c L$ to $N_a \times I$. The resulting cobordism L' (with corner rounded) clearly admits the desired structures, and a no longer occurs as orbit type in $(M \setminus N_a)\times 1$ or in $\partial_+ L$. Thus L' is a cobordism of M to $\partial_+ L$ representing a class in $I_m^G(H;A')$.
(β,γ) *is exact* Starting with M as above we form N_a, then ∂N_a. But this bounds the complement of N_a in M, so represents zero in $I_{m-1}^G(H;A')$. Conversely, given N with ∂N bounding C, we attach N to C along the boundary to obtain a closed manifold M, and the orbit type a occurs in M only at the centre of N.
(γ,α) *is exact* Starting with $\pi\colon N \to W$, we need only observe that ∂N bounds N to check that $\alpha\gamma = 0$. The converse is perhaps the most interesting part of exactness. If V represents an element of the kernel of $I_{m-1}^G(H;A') \to I_{m-1}^G(H;A)$, it bounds a manifold M, say. Since a is not an orbit type of $V = \partial M$, we can perform our construction in the usual way to obtain W_a and N_a in M. The complement of N now gives a cobordism of V to ∂N_a, as required. The exact sequence is thus established. □

To complete our programme, we must give some means of calculation of the group $A_m^G(H;A',a)$. We first observe that given a representative $\pi\colon N^m \to W$, we have for each $P \in W$ an induced orthogonal representation ρ of H_P on the fibre. As all isotropy groups are conjugate, we have an orthogonal representation of H_P defined for each $P \in W$. Clearly, these vary continuously with P. But since H_P is compact, neighbouring representations are conjugate. Thus each connected component of W corresponds to a single conjugacy class of representations ρ of H_P.

Now it is clear that ρ can occur if and only if each isotropy group of $\rho(\subseteq H_P \subseteq H)$ has class belonging to A', except for the isotropy group of the origin. We call such ρ (A', a)-allowable.

Since the same decomposition applies to cobordism, we find that $A_m^G(H; A', a)$ is expressed as a direct sum over allowable representations ρ of H_P (of rank $\leqq m$): say

$$A_m^G(H; A', a) = \oplus_\rho A_m^G(H; A', a, \rho).$$

Thus we are reduced to calculating the A-group for a fixed allowable representation ρ. Here, we follow the method of 7.3.

Let q be the rank of ρ. Let P be the principal $\mathbf{O}_q$-bundle associated to π. On P we have the natural action of $\mathbf{O}_q$, also an induced action of H which commutes with it, hence an action of $H \times \mathbf{O}_q$. This action (as is easily seen) has only a single orbit type, with M (say) as an isotropy group. We now use a standard method for reducing this action to a free one, to which we can apply the bundle classification theorem. In fact, let Q be the submanifold of P consisting of points with isotropy group equal (not merely conjugate) to M. Then the normaliser $N(M)$ of M in $H \times \mathbf{O}_q$ acts on Q, via a free action of $L = N(M)/M$.

In the present case, we can be even more explicit. Since P is the set of isometries of $\mathbb{R}^q$ on fibres of π, each element of P determines an explicit orthogonal representation of the stabiliser of the corresponding fibre. Fix a particular $H_P \in a$ and representation ρ of H_P in the desired equivalence class, and let Q be the subset of P inducing the representation ρ (not merely some conjugate) of the subgroup H_P. Then M is the set of elements $\{(h^{-1}, \rho(h)) \colon h \in H_P\}$ in $H \times \mathbf{O}_q$, and

$$N(M) = \{(n, r) \colon \rho(n^{-1}hn) = r^{-1}\rho(h)r \quad \text{for all} \quad h \in H_P\}$$

is an extension of the centraliser C_ρ of $\rho(H_P)$ in $\mathbf{O}_q$ by the subgroup of N_P which takes the representation ρ of H_P into some conjugate (this will in any case contain the component of the identity in N_P). We write L_ρ for $N(M)/M$, and X for Q/L_ρ. the dimension of L_ρ will depend on properties of ρ; however, we see at once that

$$x = \dim X = \dim W - \dim H + \dim H_P.$$

Also, W is determined by the closed manifold X^x, and the principal L_ρ-bundle over it, which in turn is determined by the classifying map $X \to BL_\rho$.

Theorem 7.5. *Let φ be an (A', a)-allowable representation. Write $c = \dim H - \dim a$. Then*

$$A_m^{\mathbf{O}}(H; A', a, \rho) \approx \Omega_{m-c}^{\mathbf{O}}(BL_\rho)$$

Proof. For, as was just pointed out, if the G-structure is ignored, the homotopy class of $X \to BL_\rho$ determines the isomorphism class of W with all its structure.

Since the identical argument applies to bounded manifolds, we can pass to cobordism classes. □

It is not at present clear how to modify the above to take account of G-structure.