Spaces of smooth embeddings, disjunction and surgery

Thomas G. Goodwillie, John R. Klein, Michael S. Weiss

Abstract. We describe progress in the theory of smooth embeddings over more than 50 years, starting with Whitney's embedding theorem, continuing with the generalized Whitney tricks of Haefliger and Dax, early disjunction results for embeddings due to Hatcher and Quinn, the surgery methods for constructing embeddings due to Browder and Levine, respectively, moving on to a systematic theory of multiple disjunction which builds on all the foregoing, and concluding with a functor calculus approach which reformulates the main theorem on multiple disjunction as a convergence theorem. Convergence takes place when the codimension is at least 3, giving a decomposition of the space of embeddings under scrutiny into 'homogeneous layers' which admit an attractive combinatorial description. The divergent cases are not devoid of interest, since they suggest a view of low–dimensional topology as a 'divergent' analogue of high–dimensional topology.

Section headings

1991 *Mathematics Subject Classification.* Primary 57R40, 57R65; secondary 57R42.
Key words and phrases. Embedding, disjunction, surgery, functor calculus, Poincaré duality, Eilenberg–Moore spectral sequence, Vassiliev invariants.

0. Preliminaries

0.1. Overview

This survey traces the development, over more than 50 years, of a theory of smooth embeddings resting today on two pillars: the methods of disjunction and surgery. More precisely, the theory is about homotopical and homological properties of spaces of smooth embeddings $\mathrm{emb}(M^m, N^n)$. It is more satisfactory when $n - m \geq 3$, but has something to offer in the other cases, too.

Chapter 1 is about embeddings in the metastable range, $m < 2n/3$ approximately, and the idea of producing an embedding $M \to N$ by starting with an immersion and removing self–intersections. This goes back to Whitney [Wh2], of course, and was pursued further by Haefliger [Hae1], [Hae2], Dax [Da], and Hatcher–Quinn [HaQ]. In the process, two important new insights emerged. The first of these [Hae2] is that embeddings in the metastable range are determined up to isotopy by their local behavior. However, this is only true with an unusual definition of *local* where the loci are small tubular neighborhoods of subsets of M of cardinality 1 or 2. The second insight [HaQ] is that, in the metastable range, practically any method for disjunction (here: removing mutual intersections of two embedded manifolds in a third by subjecting the embedded manifolds to isotopies) can serve as a method for removing self–intersections of one manifold in another.

Chapter 2, about surgery methods for constructing embeddings of M in N, gives about equal weight to the Browder approach [Br2], which is to start with an embedding $M \to N'$ and a degree one normal map $N' \to N$, and the slightly older Levine approach [Lev], which is to start with a degree one normal map $M' \to M$ and an embedding $M' \to N$. The Browder approach leads eventually to the Browder–Casson–Sullivan–Wall theorem which, assuming $n - m \geq 3$ and $n \geq 5$, essentially expresses the block embedding space $\mathrm{emb}^{\sim}(M, N)$, a rough approximation to $\mathrm{emb}(M, N)$, in terms of the space of Poincaré duality (block) embeddings from M to N. The Levine approach does not give such a neat reduction, but in contrast to the Browder approach it does lead to some ideas on how to construct embeddings of one Poincaré duality space in another. These ideas inspired work by Williams [Wi], Richter [Ric], and more recently by Klein [Kl1], [Kl2], [Kl3], which is summarized in the later parts of chapter 2.

Chapter 3 is a systematic account of multiple disjunction alias higher excision (here: an obstruction theory for making a finite number of submanifolds $M_i \subset N$ pairwise disjoint by subjecting them to isotopies in N). The most difficult ingredient is [Go1], a multiple disjunction theorem for smooth concordance embeddings (concordances alias pseudo–isotopies from a fixed smooth embedding $f_0 \colon M \to N$ to a variable one, $f_1 \colon M \to N$). Another important ingredient is a multiple disjunction theorem for (spaces of) Poincaré embeddings [GoKl], which uses [Go6] and some of the results described at the end of chapter 2. Via the Browder–Casson–Sullivan–Wall theorem, this leads to a disjunction theorem for block embedding spaces, which combines well with the aforementioned multiple disjunction theorem for concordance embeddings, resulting in a multiple disjunction theorem for honest embeddings. See [Go7].

In chapter 4, we take up and develop further Haefliger's localization ideas described in chapter 1. Specifically, we construct a sequence of approximations $T_k\,\mathrm{emb}(M, N)$ to $\mathrm{emb}(M, N)$. A point in $T_k\,\mathrm{emb}(M, N)$ is a coherent family of embeddings $V \to N$, where V runs through the tubular neighborhoods of subsets of M of cardinality $\leq k$; in particular, $T_2\,\mathrm{emb}(M, N)$ is Haefliger's approximation to $\mathrm{emb}(M, N)$, and $T_1\,\mathrm{emb}(M, N)$ is homotopy equivalent to the space of smooth immersions from $M \to N$, if $m < n$. Just as Hatcher–Quinn disjunction can be used to prove that the Haefliger approximation is a good one, so the higher disjunction results of chapter 3 are used to show that the approximations $T_k\,\mathrm{emb}(M, N)$ converge to $\mathrm{emb}(M, N)$ as $k \to \infty$, provided $n - m \geq 3$. Actually, in the cases when $2m < n - 2$, only a very easy result from chapter 3 is used. In all cases, the relative homotopy of the forgetful maps $T_k\,\mathrm{emb}(M, N) \to T_{k-1}\,\mathrm{emb}(M, N)$ is fairly manageable.

Chapter 5 applies the same localization ideas to the (generalized) homology of $\mathrm{emb}(M, N)$. What we get turns out to be a generalization of the generalization due to Rector [Re] and Bousfield [Bou] of the Eilenberg–Moore spectral sequence [EM]. The convergence issue is more complex in this case, but we have a satisfactory result for the cases where $n > 2m+1$. For $m = 1$ and $n \geq 3$ we make a connection with the Vassiliev theory of knot invariants [Va1], [Va2], [Va3], [BaN], [BaNSt], [Ko].

0.2. Notation, Terminology

Sets. Given a set X and $x \in X$, we often write x for the subset $\{x\}$. In particular, if $x_1, x_2, \ldots$ are elements of X and $f\colon X \to Y$ is any map, we may write $f|x_1$ and $f|x_1 \cup x_2$ etc. for the restrictions of f to $\{x_1\}$, $\{x_1, x_2\}$ etc. .

Spaces. All spaces in sight are understood to be compactly generated weak Hausdorff. (A space X is compactly generated weak Hausdorff if and only if the canonical map $\operatorname{colim}_{K \subset X} K \to X$, with K ranging over the compact Hausdorff subspaces of X, is a homeomorphism). Products and mapping spaces are formed in the category of such spaces in the usual way, and are related by adjunction. Pointed spaces (alias based spaces) are understood to have nondegenerate basepoints.

As is customary, we write QX for $\Omega^\infty \Sigma^\infty X$ where X is a based space, and $Q(X_+)$ or $Q_+(X)$ for $\Omega^\infty \Sigma^\infty (X_+)$ where X is unbased. Occasionally we will need a twisted version of $Q_+(X)$, as follows. Suppose that X is finite dimensional, and equipped with two real vector bundles ζ and ξ. Choose a vector bundle monomorphism $\xi \to \varepsilon^i$ where ε^i is a trivial vector bundle on X. Let

$$Q_+(X; \zeta - \xi)$$

be $\Omega^i Q$ of the Thom space of the vector bundle $\zeta \oplus \varepsilon^i/\xi$ on X. This is essentially independent of the choice of vector bundle monomorphism $\xi \to \varepsilon$ made. We will also use this notation when X is infinite dimensional, and the bundle ξ is in some obvious way pulled back from a finite dimensional space.

More generally, with X, ζ, ξ as before and $A \subset X$ a closed subset for which the inclusion is a cofibration, we let

$$Q(X/A; \zeta - \xi)$$

be $\Omega^i Q$ of a certain quotient of Thom spaces (Thom space of $\zeta \oplus \varepsilon^i/\xi$ on X, modulo Thom space of the restriction of $\zeta \oplus \varepsilon^i/\xi$ to A).

Cubical diagrams. Let S be a finite set. An S–cube of spaces is a covariant functor $R \mapsto X(R)$ from the poset of subsets of S to spaces. It is *k–cartesian* if the canonical map (whose homotopy fibers are the *total homotopy fibers* of X)

$$X(\emptyset) \longrightarrow \operatorname*{holim}_{R\neq\emptyset} X(R)$$

is k–connected. (Here the homotopy inverse limit can be described explicitly as the space of natural transformations from $R \mapsto \Delta(R)$ to $R \mapsto X(R)$, where $\Delta(R)$ is the simplex of dimension $|R|-1$ spanned by R, assuming $R \neq \emptyset$.) The cube is *k–cocartesian* if the canonical map (whose homotopy cofiber is the *total homotopy cofiber* of X)

$$\operatorname*{hocolim}_{R\neq S} X(R) \longrightarrow X(S)$$

is k–connected. (Here the homotopy colimit can be described explicitly as the quotient of $\coprod_{R\neq S} \Delta(S\smallsetminus R)\times X(R)$ by relations $(i_*a, b) \simeq (a, i_*b)$ where $i\colon R_1 \to R_2$ is an inclusion of proper subsets of S.) In both cases, $k = \infty$ is allowed. If X is a functor from the poset of subsets of S to pointed spaces, then the canonical map $X(\emptyset) \to \operatorname{holim}_{R\neq\emptyset} X(R)$ is pointed; its homotopy fiber over the base point will be called *the* total homotopy fiber of X.

The poset of subsets of S is isomorphic to its own opposite, so we use similar language for contravariant functors from it to spaces.

An S–cube is *strongly ∞–cocartesian* if all its 2–dimensional subcubes are ∞–cocartesian, and *strongly ∞–cartesian* if all its 2–dimensional subcubes are ∞–cartesian. For $|S| \geq 2$, strongly ∞–cocartesian/cartesian implies ∞–cocartesian/cartesian.

A contravariant S–cube X of spaces in which $S = \{1, \ldots, n-1\}$ is called an *n–ad* if the maps from $X(R)$ to $X(\emptyset)$ are inclusions, for any $R \subset S$, and $X(R) = \bigcap_{i\in R} X(i) \subset X_\emptyset$. The n–ad is *special* if $X(S) = \emptyset$. The n–ad is a *manifold* n–ad if each $X(R)$ is a manifold with boundary $\bigcup_{i\notin R} X(R \cup i)$. In the smooth setting, each $X(R)$ is required to be a smooth manifold with appropriate corners in the boundary.

Homotopy (co–)limits. For homotopy limits and homotopy colimits in general, see [BK]. We like the point of view of [Dr] and [DwK2], which is as follows, in outline. A functor E from a small category $\mathcal{C}$ to spaces is a *CW–functor* if it is a monotone union of subfunctors E_{-1} , E_0 , E_1 , E_2 , $\ldots$, where E_i has been obtained from E_{i-1} by attachment of so–called i–cells. These are functors of the form $c \mapsto \mathbb{D}^i \times \operatorname{mor}_{\mathcal{C}}(c, d)$, for some d in $\mathcal{C}$. Every functor F from $\mathcal{C}$ to spaces has a CW–approximation $F^\sim \to F$ (in which $F^\sim$ is a CW–functor, and $F^\sim \to F$ specializes to weak homotopy equivalences $F^\sim(c) \to F(c)$ for each c in $\mathcal{C}$). Put

$$\text{hocolim } F := \text{colim } F^\sim\,, \qquad \text{holim } F := \text{nat}((*_{\mathcal{C}})^\sim, F)$$

where nat denotes the space of natural transformations and $*_{\mathcal{C}}$ is the constant functor $c \mapsto *$ on $\mathcal{C}$. For colimits, see [MaL]. For more on homotopy limits and homotopy colimits, see also [DwK1].

Manifolds. All manifolds in this survey are assumed to have a countable base for their topology. Manifolds are without boundary unless otherwise stated; a manifold *with boundary* may of course have empty boundary.

We write $\text{emb}(M,N)$ for the space of smooth embeddings from M to N, and $\text{imm}(M,N)$ for the space of smooth immersions, both defined as geometric realizations of certain simplicial sets. Unless otherwise stated, M and N are assumed to be smooth without boundary.

Let G be a finite group. A map $f\colon K \to L$ of manifolds with G–action is *equivariant* if it is a G–map, and *isovariant* if, in addition, $f^{-1}(L^H) = K^H$ for every subgroup $H \le G$. If K, L are smooth and f is a smooth map, it is natural to combine isovariance as above with "infinitesimal" isovariance: call f *strongly isovariant* if it is isovariant and, for each $H \le G$ and $x \in K^H$, the differential $T_x f$ of f at x is an isovariant linear map from $T_x K$ to $T_{f(x)} L$.

Poincaré spaces. Poincaré space is short for simple Poincaré duality space, alias simple Poincaré complex [Wa2, 2nd ed., §2]; Poincaré pair is short for simple Poincaré duality pair. The fundamental class $[X]$ of a Poincaré space X of formal dimension n lives in $H_n(X;\mathbb{Z}^t)$, where $\mathbb{Z}^t$ denotes a local coefficient system on X with fibers isomorphic to $\mathbb{Z}$. Together, $[X]$ and $\mathbb{Z}^t$ are determined by X, up to a unique isomorphism between local coefficient systems on X.

What is more, there exist a fibration ν^k on X with fibers $\simeq \mathbb{S}^{k-1}$, and a 'degree one' map ρ from $\mathbb{S}^{n+k}$ to the Thom space (mapping cone) of ν ; together, ν and ρ are unique up to contractible choice if k is allowed to tend to ∞. See [Br3], [Ra]. The fibration ν is known as the Spivak normal fibration of X. The image of $[\rho]$ under Hurewicz homomorphism and Thom isomorphism is a fundamental class in $H_n(X;\mathbb{Z}^t)$ where $\mathbb{Z}^t$ is the twisted integer coefficient system associated with ν. Something analogous is true for Poincaré pairs.

1. Double point obstructions

1.1. The Whitney embedding theorem

1.1.1. Theorem [Wh2]. *For $m > 0$, every smooth m–manifold M can be embedded in $\mathbb{R}^{2m}$.*

Whitney's proof of 1.1.1 relies on the fact [Wh1] that M^m can be immersed in $\mathbb{R}^{2m}$. He also knew [Wh1] that any immersion $M^m \to \mathbb{R}^{2m}$ can be approximated by one with transverse self–intersections. The other main ideas are these:

(i) Without loss of generality, M^m is connected. Suppose that M is also closed. Then any immersion $f\colon M^m \to \mathbb{R}^{2m}$ has an algebraic self–intersection number I_f (to be defined below) which is an integer if m is even and orientable, otherwise an integer modulo 2.

(ii) (*Whitney trick*) In the situation of (i), the immersion f is regularly homotopic to an immersion with exactly $|I_f|$ transverse self–intersections (and no other self–intersections), provided $m > 2$. Here $|I_f|$ should be read as 0 or 1 if $I_f \in \mathbb{Z}/2$.

(iii) For every $m > 0$, there exists an immersion $g\colon \mathbb{S}^m \longrightarrow \mathbb{R}^{2m}$ having algebraic self–intersection number $I_g = 1$.

Assuming (i), (ii), (iii), the proof of 1.1.1 for $m > 2$ is completed as follows. We start by choosing some immersion $f_0\colon M^m \to \mathbb{R}^{2m}$. In the closed connected case, we use (iii) to modify it, so that an immersion $f\colon M \to \mathbb{R}^{2m}$ with $I_f = 0$ results. Then (ii) can be applied. In the case where M is open and connected, and all self–intersections are transverse, it is easy to "indent" M appropriately, i.e. to find an embedding $e\colon M \to M$ isotopic to the identity such that $f := f_0 e$ is an embedding. See [Wh2, §8] for details.

1.1.2. Definitions. Whitney gives two definitions of I_f. For the first, assume that $f\colon M \to \mathbb{R}^{2m}$ is an immersion with transverse self–intersections only. Count the self–intersections (with appropriate sign ± 1 if m is even and M is orientable, otherwise modulo 2). The result is I_f.

For the second definition, let $f\colon M \to \mathbb{R}^{2m}$ be any immersion. Define

$$\beta\colon M \times M \setminus \Delta_M \longrightarrow \mathbb{R}^{2m}$$

by $\beta(x,y) := f(x) - f(y)$. Then $\beta^{-1}(0)$ is compact and β is $\mathbb{Z}/2$–equivariant, where the generator of $\mathbb{Z}/2$ acts on the domain (freely) and codomain (not freely) by $(x,y) \mapsto (y,x)$ and by $z \mapsto -z$, respectively.

Hence β has a well defined degree I_f in $\mathbb{Z}$ or $\mathbb{Z}/2$. It can be found by deforming β in a small neighborhood of $\beta^{-1}(0)$ so that β becomes transverse to 0, and counting $\mathbb{Z}/2$–orbits in the inverse image of 0 (with appropriate signs when m is even and M is orientable).

We assume that statements (ii) and (iii) above are well known through [Mi1]. We will see plenty of generalizations quite soon.

Remark. Whitney's I_f has precursors in [van]. See also [Sha].

1.2. Scanning

The theorem of Haefliger that we are about to present dates back to the early sixties. The immersion classification theorem was available [Sm1], [Hi1]; see also [Hae3]. It states that if M^m and N^n are smooth, $m < n$, or $m = n$ and M open, then an evident map from $\mathrm{imm}(M,N)$ to the space of pairs (f,g), with $f\colon M \to N$ continuous and $g\colon \tau_M \to f^*\tau_N$ fiberwise monomorphic (and linear), is a (weak) homotopy equivalence. In addition, transversality concepts had conquered differential topology. In particular, it was known that a "generic" smooth immersion $M^m \to N^n$ would have transverse self–intersections only, of multiplicity $\le n/(n-m)$. It was therefore natural for Haefliger to impose the condition $n/(n-m) < 3$, equivalently $m < 2n/3$ (*metastable range*), which ensures that all self–intersection points in a generic immersion $M \to N$ are double points, and to view an embedding $M \to N$ as an immersion without double points.

Notation. In 1.2.1 below we write $\mathrm{map}(\dots)$, $\mathrm{map}^G(\dots)$, $\mathrm{ivmap}^G(\dots)$ for spaces of smooth maps, equivariant smooth maps, strongly isovariant smooth maps, respectively, all to be defined as geometric realizations of simplicial sets.

1.2.1. Theorem [Hae2]**.** *If $m+1 < 2n/3$, then the following square is 1–cartesian :*

$$\begin{array}{ccc}
\mathrm{emb}(M,N) & \xrightarrow{\subset} & \mathrm{map}(M,N) \\
\big\downarrow & & \big\downarrow{\scriptstyle f\mapsto f\times f} \\
\mathrm{ivmap}^{\mathbb{Z}/2}(M\times M, N\times N) & \xrightarrow{\subset} & \mathrm{map}^{\mathbb{Z}/2}(M\times M, N\times N)\,.
\end{array}$$

Remark. Haefliger's original statement is slightly different: in his definitions of the mapping spaces involved, other than $\mathrm{emb}(M,N)$, he does not ask for *smooth* maps. A vector bundle theoretic argument [HaeH, 4.3.a] shows that the two versions are equivalent.

It will turn out that the square in 1.2.1 is $(2n-3-3m)$–cartesian, an improvement which is essentially due to Dax [Da]. We will sketch the proof in section 1.3, and again in section 1.4, following Dax more closely.

1.2.2. Example [Hae2]**.** Let $N = \mathbb{R}^n$. Then $\mathrm{map}(M, N)$ is contractible and so is $\mathrm{map}^{\mathbb{Z}/2}(M \times M, N \times N) \cong \mathrm{map}(M \times M, N)$. Therefore 1.2.1 implies that

$$\mathrm{emb}(M, \mathbb{R}^n) \longrightarrow \mathrm{ivmap}^{\mathbb{Z}/2}(M \times M, \mathbb{R}^n \times \mathbb{R}^n)$$

given by $f \mapsto f \times f$ is 1–connected, if $m+1 < 2n/3$. Now an isovariant map g from $M \times M$ to $\mathbb{R}^n \times \mathbb{R}^n$ determines an equivariant map vgj from $M \times M \smallsetminus \Delta_M$ to $\mathbb{S}^{n-1}$, where $j\colon M \times M \smallsetminus \Delta_M \to M \times M$ is the inclusion and v is the map $(x, y) \mapsto (x-y)/|x-y|$ from $\mathbb{R}^n \times \mathbb{R}^n$ minus diagonal to $\mathbb{S}^{n-1}$. It follows easily from [HaeH, 4.3.a] that $g \mapsto vgj$ is 1–connected if $m+1 < 2n/3$. Hence isotopy classes of smooth embeddings of M^m in $\mathbb{R}^n$, for $m+1 < 2n/3$, are in bijection with homotopy classes of equivariant maps from $M \times M \smallsetminus \Delta_M$ to $\mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1}$ is equipped with the antipodal action of $\mathbb{Z}/2$.

We now briefly justify our use of the word *scanning* in the title of this subsection. The upper horizontal map in the diagram in 1.2.1 captures, for each $f \in \mathrm{emb}(M, N)$, the restricted embeddings $f|S$ where S runs through the one–element subsets of M. The left–hand vertical map captures, for each $f \in \mathrm{emb}(M, N)$, the restricted embeddings $f|S$ where S runs through the 2–element subsets of M (the two elements are allowed to 'collide'); it also captures the tangent bundle monomorphism induced by f. The remaining two arrows capture coherence.

1.3. Disjunction

Disjunction theory, as we understand it here, is about the elimination of intersections of two or more manifolds, each embedded in a common ambient manifold, by means of isotopies of the embedded manifolds. Families of such elimination problems are also considered. An important theme is that disjunction homotopies can often be improved to disjunction isotopies, as in the following theorem.

1.3.1. Theorem. *Let L^ℓ, M^m, N^n be smooth, L and M closed, L contained in N as a smooth submanifold. The following square of inclusion* *maps is* $(2n-3-2m-\ell)$*–cartesian:*

$$\begin{array}{ccc} \mathrm{emb}(M, N \smallsetminus L) & \longrightarrow & \mathrm{emb}(M,N) \\ \big\downarrow & & \big\downarrow \\ \mathrm{map}(M, N \smallsetminus L) & \longrightarrow & \mathrm{map}(M,N). \end{array}$$

Idea of proof. Let $\{h_t\colon M \times \Delta^k \to N \mid 0 \le t \le 1\}$ be a smooth homotopy such that $h_0|M \times y$ is an embedding and $h_1|M \times y$ has image in $N \smallsetminus L$, for all $y \in \Delta^k$. Suppose also that $h = \{h_t\}$ is a constant homotopy on $M \times \partial\Delta^k$. Let $Z \subset M \times \Delta^k \times [0,1]$ consist of all points (x,y,t) such that $h_s|M \times y$ is singular at $x \in M$ for some $s \le t$. If $h = \{h_t\}$ is 'generic' and k is not too large, for example $k \le (2n-3-2m-\ell)-1$, then Z will have empty intersection with $h^{-1}(L)$. Then it is easy to find a smooth function $\psi\colon M \times \Delta^k \to [0,1]$ such that Z lies above the graph of ψ, and $h^{-1}(L)$ lies below it. Using this, one deforms h to the homotopy $h^!$ given by $h^!_t(x,y) = h_{\psi(x)t}(x,y)$. Now $h^!$ is adjoint to a homotopy of maps $\Delta^k \to \mathrm{emb}(M,N)$ and $h^!_1$ is adjoint to a map $\Delta^k \to \mathrm{emb}(M, N \smallsetminus L)$. This shows that the square in 1.3.1 is k–cartesian, with $k = (2n-3-2m-\ell)-1$. A little extra work improves the estimate to $2n-3-2m-\ell$. $\square$

Earlier results in the direction of 1.3.1 can be found in [Sta], [Wa1], [Lau1], [Ti], [Lau2] and [Lau3]. The method of proof is a simple example of *sunny collapsing*, an idea which appears to originate in Zeeman's PL unknotting work [Ze]; see also [Hu1].

1.3.2. Corollary. *Let* L^ℓ, M^m, N^n *be smooth,* L *and* M *closed,* L *and* M *contained in* N *as smooth submanifolds. The homotopy fiber of*

$$\mathrm{emb}(L \amalg M, N) \to \mathrm{emb}(L,N) \times \mathrm{emb}(M,N)$$

has a $\min\{2n-2m-\ell-3, 2n-2\ell-m-3\}$*–connected scanning map to the section space* $\Gamma(u)$ *, where* u *is a fibration over* $M \times L$ *with fiber over* (x,y) *equal to the homotopy fiber of*

$$\mathrm{emb}(x \amalg y, N) \to \mathrm{emb}(x,N) \times \mathrm{emb}(y,N).$$

Proof. We use a Fubini type argument. First, scan along M. The homotopy fiber of $\mathrm{emb}(L \amalg M, N) \to \mathrm{emb}(L,N) \times \mathrm{emb}(M,N)$ is homotopy equivalent to the homotopy fiber of $\mathrm{emb}(M, N \smallsetminus L) \hookrightarrow \mathrm{emb}(M,N)$. The homotopy fiber of $\mathrm{emb}(L \amalg x, N) \longrightarrow \mathrm{emb}(L, N) \times \mathrm{emb}(x, N)$ is homotopy equivalent to the homotopy fiber of $\mathrm{emb}(x, N \smallsetminus L) \hookrightarrow \mathrm{emb}(x, N)$, for every $x \in M$. So by 1.3.1, scanning along M gives a $(2n-2m-\ell-3)$–connected map from the homotopy fiber of $\mathrm{emb}(L \amalg M, N) \to \mathrm{emb}(L, N) \times \mathrm{emb}(M, N)$ to $\Gamma(v)$, where v is a fibration on M whose fiber over $x \in M$ is the homotopy fiber of $\mathrm{emb}(L \amalg x, N) \longrightarrow \mathrm{emb}(L, N) \times \mathrm{emb}(x, N)$.

We get from $\Gamma(v)$ to $\Gamma(u)$ by scanning along L. Note that for each x in M, the homotopy fiber of $\mathrm{emb}(L \amalg x, N) \longrightarrow \mathrm{emb}(L, N) \times \mathrm{emb}(x, N)$ is homotopy equivalent to the homotopy fiber of the inclusion of $\mathrm{emb}(L, N \smallsetminus x)$ in $\mathrm{emb}(L, N)$. Therefore another application of 1.3.1 shows that our second scanning map is $((2n-2\ell-0-3)-m)$–connected. Hence the composite scanning map is $\min\{2n-2m-\ell-3, 2n-2\ell-m-3\}$–connected. $\square$

Terminology. Eventually we will need a relative version of 1.3.2. In the most general relative version, N is a manifold with boundary, and L, M are compact triads. For L this means that ∂L is the union of smooth codimension zero submanifolds $\partial_0 L$ and $\partial_1 L$ with

$$\partial\partial_0 L = \partial\partial_1 L = \partial_0 L \cap \partial_1 L\,.$$

L is viewed as a manifold with corners (corner set $\partial_0 L \cap \partial_1 L$). We assume that L is contained in N in such a way that $\partial_0 L = L \cap \partial N$ and the inclusion $\partial_1 L \hookrightarrow N$ is transverse to ∂N. We make analogous assumptions for M and the inclusion $M \to N$. In addition, we assume that $\partial_0 M$ and $\partial_0 L$ are disjoint, and allow only embeddings $M \to N$ and $L \to N$ which agree with the inclusions on $\partial_0 M$ and $\partial_0 L$ respectively. The appropriate section space $\Gamma(u)$ consists of sections of a certain fibration on $M \times L$ as before, but the sections are prescribed on $(\partial_0 M \times L) \cup (M \times \partial_0 L)$.

1.3.3. Corollary. *The square in* 1.2.1 *is* $(2n-3-3m)$ *—connected.*

Proof, in outline. Let $\mathrm{emb}_h(M, N)$ be the Haefliger approximation to $\mathrm{emb}(M, N)$. That is, $\mathrm{emb}_h(M, N)$ is the homotopy pullback of the lower left hand, upper right hand and lower right hand terms in 1.2.1. We have to show that Haefliger's map

$$\mathrm{emb}(M, N) \to \mathrm{emb}_h(M, N)$$

is $(2n-3-3m)$–connected. It suffices to establish this in the case where $M = \bar{M} \smallsetminus \partial\bar{M}$ for a compact smooth manifold $\bar{M}$ with boundary. We can suppose that $\bar{M}$ comes with a handle decomposition. More specifically, suppose the handles are all of index $\le r$, and the number of handles of index r is a_r. We proceed by induction on r, and for fixed r we proceed by induction on a_r.

Choose a handle $H \cong \mathbb{D}^r \times \mathbb{D}^{m-r}$ in M of maximal index r. If $r = 0$ there is not much to prove, so we assume $r > 0$. We can then choose two disjoint index r subhandles H_1 and H_2 of H. (In the coordinates for H, these would correspond to $C_1 \times \mathbb{D}^{m-r}$ and $C_2 \times \mathbb{D}^{m-r}$ where C_1 and C_2 are small disjoint disks in $\mathbb{D}^r$.)

Let $M_i = M \smallsetminus H_i$ for $i = 1, 2$, and $M_T = \cap_{i \in T} M_i$ for $T \subset \{1, 2\}$. For $T \neq \emptyset$, the closure of M_T in $\bar{M}$ has a handle decomposition with fewer than a_r handles of index r, and no handles of index $> r$. By induction, $\mathrm{emb}(M_T, N) \to \mathrm{emb}_h(M_T, N)$ is $(2n - 3 - 3m)$–connected for $T \neq \emptyset$.

The spaces $\mathrm{emb}(M_T, N)$ and the restriction maps between them form a commutative square, denoted $\mathrm{emb}(M_\bullet, N)$. We have another commutative square $\mathrm{emb}_h(M_\bullet, N)$ and a Haefliger map

$$\mathrm{emb}(M_\bullet, N) \longrightarrow \mathrm{emb}_h(M_\bullet, N)\,.$$

Looking at the induced map from any of the total homotopy fibers of $\mathrm{emb}(M_\bullet, N)$ to the corresponding homotopy fiber of $\mathrm{emb}_h(M_\bullet, N)$, one finds that it is an instance of scanning essentially as in 1.3.2 (see the details just below). By 1.3.4, it is $(2n - 3m - 3)$–connected. Combined with the inductive assumption, that the map $\mathrm{emb}(M_T, N) \longrightarrow \mathrm{emb}_h(M_T, N)$ is $(2n - 3m - 3)$–connected for $T \neq \emptyset$, this shows that Haefliger's map $\mathrm{emb}(M_T, N) \longrightarrow \mathrm{emb}_h(M_T, N)$ is also $(2n - 3m - 3)$–connected when T is empty. $\square$

Details. To understand the total homotopy fibers of $\mathrm{emb}(M_\bullet, N)$ in the above proof, replace $\mathrm{emb}(M_\bullet, N)$ by $\mathrm{emb}(\bar{M}_\bullet, N)$ where $\bar{M}_T$ is the closure of M_T in $\bar{M}$. (Our notation $\mathrm{emb}(\bar{M}_T, N)$ is legalized by the remark just before 1.3.3, provided we decree $\partial_0 \bar{M}_T = \emptyset$.) By the isotopy extension theorem, all maps in $\mathrm{emb}(\bar{M}_\bullet, N)$ are fibrations. Hence we can obtain all total homotopy fibers as homotopy fibers of subsquares of the form $\mathrm{emb}(\bar{M}_\bullet, N; g)$, where $g\colon \bar{M}_{\{1,2\}} \to N$ is an embedding and $\mathrm{emb}(\bar{M}_T, N; g)$ denotes the space of embeddings $\bar{M}_T \to N$ extending g. Modulo natural homotopy equivalences, these subsquares can be rewritten in the form $\mathrm{emb}(H_\bullet, N_g)$ where $H_T = \cup_{i \in T} H_i$ for $T \subset \{1, 2\}$ and $N_g \subset N$ is the closure of the complement of a thickening of $\mathrm{im}(g)$ in N. Boundary conditions are understood: $\partial_0 H_i = H_i \cap \bar{M}_i$. With that, we are in the situation of 1.3.2 (relative version) and obtain a scanning map to a section space $\Gamma(u_g)$, where u_g is a fibration on $H_1 \times H_2$.

We can also recast the relevant total homotopy fibers of the square $\mathrm{emb}_h(\bar{M}_\bullet, N)$ as total homotopy fibers of subsquares $\mathrm{emb}_h(\bar{M}_\bullet, N; g)$ with g as before. Here $\mathrm{emb}_h(\bar{M}_T, N; g)$ is the fiber of $\mathrm{emb}_h(\bar{M}_T, N)$ over the element of $\mathrm{emb}_h(\bar{M}_{\{1,2\}}, N)$ determined by the embedding g. There is a scanning map (which is a homotopy equivalence) from the total homotopy fiber of $\mathrm{emb}_h(\bar{M}_\bullet, N; g)$ to a section space $\Gamma(v_g)$. The sections are subject to boundary conditions as usual. Again v_g is a fibration on $H_1 \times H_2$, containing u_g. The fiber of v_g over $(x,y) \in H_1 \times H_2$ is

$$\begin{aligned} &\mathrm{hofiber}\,[\,\mathrm{emb}_h(x \amalg y, N) \longrightarrow \mathrm{emb}_h(x,N) \times \mathrm{emb}_h(y,N)\,] \\ \simeq\; &\mathrm{hofiber}\,[\,\mathrm{emb}(x \amalg y, N) \longrightarrow \mathrm{emb}(x,N) \times \mathrm{emb}(y,N)\,]\,. \end{aligned}$$

The inclusion $u_g \to v_g$ is not a fiber homotopy equivalence in general (because N_g is not the same as N), but it is $(2n-m-3)$–connected on fibers. Hence the induced map $\Gamma(u_g) \to \Gamma(v_g)$ is $(2n-3m-3)$–connected. □

1.4. The stable point of view

Although Haefliger's scanning idea was a new departure, his proof of 1.2.1 used "conservative" double point elimination methods as in 1.1. About ten years later, Dax [Da] and Hatcher–Quinn [HaQ] developed the double point elimination methods into a full–blown theory, of which we want to give an idea. (See [Sa] and [LLZ] and for the analogous double point elimination approach to block embedding spaces $\mathrm{emb}^{\sim}(M,N)$, defined in 2.2 below.)

Suppose that $f \colon M \to N$ is any smooth immersion which is generic (the tangent spaces of M at self–intersection points in N are in general position). Suppose that M is closed. Let $E^{\curlyvee}(f,f)$ be the space of triples (x,y,ω) where $(x,y) \in M \times M \smallsetminus \Delta_M$ and $\omega \colon [-1,+1] \to N$ is a path from $f(x)$ to $f(y)$ in N. Think of it as a space over $M \times M \smallsetminus \Delta_M$. There is an involution on $E^{\curlyvee}(f,f)$ given by $(x,y,\omega) \mapsto (y,x,\omega^{-1})$, where ω^{-1} is ω in reverse. The projection to $M \times M \smallsetminus \Delta_M$ is equivariant. Let

$$\langle f \pitchfork f \rangle \subset E^{\curlyvee}(f,f)_{\mathbb{Z}/2}$$

consist of all (orbits of) triples (x,y,ω) in $E^{\curlyvee}(f,f)$ with constant path ω. Then $\langle f \pitchfork f \rangle$ is a smooth manifold which maps to the self–intersection set of $f(M)$ in N and should be viewed as a resolution of it. If $m < 2n/3$, then the resolving map is a diffeomorphism.

Next we discuss normal data. There are maps from $E^{\curlyvee}(f,f)$ to N and $M \times M \smallsetminus \Delta_M$ given by $(x,y,\omega) \mapsto \omega(0)$ and $(x,y,\omega) \mapsto (x,y)$, respectively, which we can use to pull back the tangent bundles τ_N and

$\tau_{M\times M}$. The maps are equivariant (trivial involution on N), so we have canonical choices κ_1 and κ_2 of involutions on the pullback bundles covering the standard involution on $E^{\curlyvee}(f,f)$. However, we use $-\operatorname{id}\cdot\kappa_1$ and κ_2 to view τ_N and $\tau_{M\times M}$, and then $\tau_N-\tau_{M\times M}$, as (virtual) vector bundles on $E^{\curlyvee}(f,f)_{\mathbb{Z}/2}$. Then we can say that the (absolute) normal bundle of $\langle f \pitchfork f\rangle$ is identified with the virtual vector bundle which is the pullback of $\tau_N-\tau_{M\times M}$ under $\langle f \pitchfork f\rangle \hookrightarrow E^{\curlyvee}(f,f)_{\mathbb{Z}/2}$. Therefore $\langle f \pitchfork f\rangle$ can be viewed as a "bordism element" or, by the Thom–Pontryagin construction, as a point in the infinite loop space

$$Q_+(E^{\curlyvee}(f,f)_{\mathbb{Z}/2}\,;\,\tau_N-\tau_{M\times M}).$$

Next, fix some γ in the homotopy fiber of $\operatorname{emb}(M,N)\to\operatorname{imm}(M,N)$ over f. We assume that γ is smooth and generic when viewed as a map from $M\times[0,1]$ to $N\times[0,1]$ over $[0,1]$; this implies that the self–intersections are transverse. Let $\langle \gamma \pitchfork \gamma\rangle\subset E^{\curlyvee}(f,f)\times[0,1]$ consist of all quadruples (x,y,ω,t) where $\gamma_t(x)=\gamma_t(y)\in N$ and ω is the path

$$s\mapsto\begin{cases}\gamma_{t-s(1-t)}(x) & -1\le s\le 0\\ \gamma_{t+s(1-t)}(y) & 0\le s\le 1.\end{cases}$$

A discussion like the one above shows that $\langle \gamma \pitchfork \gamma\rangle$ determines a path from $*$ to $\langle f \pitchfork f\rangle$ in $Q_+(E^{\curlyvee}(f,f)_{\mathbb{Z}/2}\,;\,\tau_N-\tau_{M\times M})$, via the Thom–Pontryagin construction. The procedure generalizes easily to generic families, more precisely, generic maps from some simplex Δ^k to $\phi(f)$, and in this way gives a map

$$\begin{array}{c}\text{hofiber}_f\,[\operatorname{emb}(M,N)\to\operatorname{imm}(M,N)\\ \big\downarrow\\ \text{paths from } * \text{ to } \langle f \pitchfork f\rangle \text{ in } Q_+(E^{\curlyvee}(f,f)_{\mathbb{Z}/2}\,;\,\tau_N-\tau_{M\times M})\,.\end{array}\tag{1.4.1}$$

1.4.2. Theorem [Da, VII.2.1]**;** see also [HaQ]**.** *This map is $(2n-3-3m)$–connected.*

Dax' proof of 1.4.2 is based on a "higher" Whitney trick, a purely geometric statement about the realizability of abstract nullbordisms of a self–intersection manifold (or family of such) by regular homotopies of the immersed manifold (or family of such). The higher Whitney trick is very beautifully distilled in [HaQ].

There is another proof of 1.4.2 by reduction to 1.3.3, as we now explain. The map in 1.4.2 is a composition

$$
\begin{array}{c}
\operatorname{hofiber}_f[\operatorname{emb}(M,N) \to \operatorname{imm}(M,N)] \\
\Big\downarrow{\scriptstyle \text{scanning}} \\
\Gamma_c^{\mathbb{Z}/2}(p_f) \\
\Big\downarrow \\
\text{paths from } * \text{ to } \langle f \pitchfork f\rangle \text{ in } Q_+(E^{\curlyvee}(f,f)_{\mathbb{Z}/2}\,;\,\tau_N - \tau_{M\times M})\,.
\end{array}
\tag{1.4.3}
$$

Here p_f is the fibration on $M \times M \smallsetminus \Delta_M$ whose fiber over (x,y) is the homotopy fiber of $\operatorname{emb}(x \cup y, N) \to \operatorname{imm}(x \cup y, N)$ over the point $f|x \cup y$. We say that a section s of p_f has compact support if, for every (x,y) in $M \times M$ sufficiently close to but not in Δ_M, the value $s(x,y)$ belongs to the homotopy fiber of the identity map $\operatorname{emb}(x \cup y, N) \to \operatorname{emb}(x \cup y, N)$ over the point $f|x \cup y$. (Note: $f|x \cup y$ is indeed an embedding for (x,y) close to the diagonal.) Restriction of embeddings and immersions from M to $x \cup y$ for $(x,y) \in M \times M \smallsetminus \Delta_M$ gives the first arrow in (1.4.3). The symbol Γ is for sections as usual; the subscript c is for compact support, and the superscript $\mathbb{Z}/2$ indicates that we obtain equivariant sections.

The second arrow in (1.4.3) is a stabilization map combined with Poincaré duality, compare [Go6, ch.7], which results from the following observation.

1.4.4. Observation. *The fiberwise unreduced suspension of p_f is fiberwise homotopy equivalent to the fiberwise (over $M \times M \smallsetminus \Delta_M$) Thom space of the vector bundle τ_N on $E^{\curlyvee}(f,f)$.*

Sketch proof. Fix some $x, y \in M$ with $x \neq y$. The fiber V of p_f over (x,y) is the homotopy fiber of the inclusion $\operatorname{emb}(x \cup y, N) \to \operatorname{map}(x \cup y, N)$ over the point $f|x \cup y$. Let W be the homotopy fiber of

$$\text{id: } \operatorname{map}(x \cup y, N) \to \operatorname{map}(x \cup y, N)$$

over the point $f|x \cup y$. Then $V \subset W$. Since W is contractible, the mapping cone of $V \hookrightarrow W$ can be identified with the unreduced suspension of V. But W is also a smooth Banach manifold, and $W \smallsetminus V$ is a codimension n smooth submanifold of W, homeomorphic to the space of paths in N from $f(x)$ to $f(y)$. The normal bundle of $W \smallsetminus V$ in W corresponds to the pullback of τ_N under the midpoint evaluation map. The mapping cone of the inclusion $V \to W$ is homotopy equivalent to the Thom space of the normal bundle of $W \smallsetminus V$ in W. $\square$

Now our deduction of 1.4.2 from 1.3.3. goes like this: the second arrow in (1.4.3) is $(2n-3-2m)$–connected by Freudenthal, while the first is $(2n-3-3m)$–connected by 1.3.3. $\square$

Dax has another result along the lines of 1.4.2, giving a homotopy theoretic analysis in the metastable range of the homotopy fiber of the inclusion $\mathrm{emb}(M,N) \to \mathrm{map}(M,N)$ over some $f \in \mathrm{map}(M,N)$. We can also recover this from 1.3.3. Note that our definition of p_f in (1.4.3) works for any continuous $f\colon M \to N$. In this generality it does not make sense to speak of sections of p_f with compact support, but we can speak of *tempered* sections of p_f ; a section s is tempered if, for (x,y) in $M \times M$ close to but not in the diagonal, the value $s(x,y)$ viewed as a path in $N^{\{x,y\}}$ stays close to the diagonal. Stabilization combined with Poincaré duality gets us from the space of tempered equivariant sections of p_f to

$$Q\left(\frac{E^{\curlyvee}(f,f)_{\mathbb{Z}/2}}{\mathbb{S}(TM)_{\mathbb{Z}/2}}\ ;\ \tau_N - \tau_{M\times M}\right)$$

where $\mathbb{S}(TM)$ is the total space of the unit sphere bundle associated with TM. (Regard it as a $\mathbb{Z}/2$–invariant subspace of $M \times M \smallsetminus \Delta_M$, namely, the boundary of a nice symmetric closed tubular neighborhood of Δ_M in $M \times M$. The inclusion of $\mathbb{S}(TM)$ in $M \times M \smallsetminus \Delta_M$ lifts canonically to an equivariant map from $\mathbb{S}(TM)$ to $E^{\curlyvee}(f,f)$.) Therefore the composition of scanning, fiberwise stabilization and Poincaré duality is a map

$$\begin{array}{c} \mathrm{hofiber}_f\,[\,\mathrm{emb}(M,N) \to \mathrm{map}(M,N)\,] \\ \Big\downarrow \\ \text{paths from } * \text{ to } \langle f \cap f\rangle \text{ in } Q\left(\dfrac{E^{\curlyvee}(f,f)_{\mathbb{Z}/2}}{\mathbb{S}(TM)_{\mathbb{Z}/2}}\ ;\ \tau_N - \tau_{M\times M}\right) \end{array} \tag{1.4.5}$$

Here the definition of $\langle f \cap f\rangle$ is a by–product of the stabilization process. To understand where it comes from, note that the fiberwise suspension of p_f (as in 1.4.4) has two distinguished sections, denoted $+1$ and -1. Stabilization and Poincaré duality take $+1$ to the base point by construction, but -1 becomes $\langle f \cap f\rangle$ by definition. — Arguing as we did in the proof of 1.4.2 from 1.3.3, we get:

1.4.6. Theorem [Da,VII.2.1]. *The map* (1.4.5) *is* $(2n-3-3m)$*–connected.*

Suppose that f in 1.4.6 is k–connected. Then the inclusion of $\mathbb{S}(TM)$ in $E^{\curlyvee}(f,f)$ is $\min\{k-1, m-2\}$–connected, by inspection. (It can be written as a composition $\mathbb{S}(TM) \to E^{\curlyvee}(\mathrm{id}_M, \mathrm{id}_M) \longrightarrow E^{\curlyvee}(f,f)$ in which the second arrow is clearly $(k-1)$-connected. The first arrow can be looked at as a map over M, and the fiber of $E^{\curlyvee}(\mathrm{id}_M, \mathrm{id}_M)$ over $x \in M$ is, up to homotopy equivalence, the homotopy fiber of $M \smallsetminus x \hookrightarrow M$.) This gives a corollary, essentially due to Haefliger again [Hae1]:

1.4.7. Corollary. *Let* $f\colon M \to N$ *be a* k*–connected map. Then the homotopy fiber of the inclusion* $\mathrm{emb}(M,N) \hookrightarrow \mathrm{map}(M,N)$ *over* f *is* $\min\{k-1+n-2m, n-m-2, 2n-3m-4\}$*–connected. In particular, it is nonempty when* $m+1 < 2n/3$ *and* $k > 2m-n$.

2. Surgery methods

We will be concerned with two methods which use surgery to construct smooth embeddings. The older one, initiated by Levine [Lev], aims to construct a smooth embedding $M \to N$ by making hypotheses of a homotopy theoretic nature which, via transversality, translate into a diagram

$$M \xleftarrow{g} M' \xrightarrow{e} N$$

where e is a smooth embedding and g is a degree one normal map, normal cobordant to the identity $M \to M$. The normal cobordism amounts to a finite sequence of elementary surgeries transforming $M' \cong e(M')$ into something diffeomorphic to M, and one tries to perform these surgeries as *embedded* surgeries, inside N. The other method, invented by Browder [Br1], [Br2], aims to construct a smooth embedding $M \to N$ by making hypotheses of a homotopy theoretic nature which, via transversality, translate into a diagram

$$M \xrightarrow{e} N' \xrightarrow{f} N$$

where e is a smooth embedding and f is a degree one normal map, normal cobordant to the identity. The normal cobordism amounts to a finite sequence of elementary surgeries transforming N' into something diffeomorphic to N, and one tries to perform these surgeries away from $e(M)$.

Reversing the historical order once again, we will begin with Browder's method, which reduces the problem of constructing embeddings $M \to N$ to a homotopy theoretic one. Then we will turn to Levine's method, to find that it has a lot to tell us about the homotopy theoretic problem created by Browder's method.

2.1. Smoothing Poincaré embeddings

Let $(M, \partial M)$ and $(N, \partial N)$ be Poincaré pairs, both of formal dimension n. By a (codimension zero) Poincaré embedding of $(M, \partial M)$ in $(N, \partial N)$ we mean a simple homotopy equivalence of Poincaré pairs

$$(M \amalg_{\partial M} C,\ \partial_1 C) \xrightarrow{f} (N, \partial N)$$

where $(C, \partial C)$ is a special Poincaré triad of formal dimension n (that is, a Poincaré pair with $\partial C = \partial_0 C \amalg \partial_1 C$) and $\partial_0 C$ is identified with ∂M. We call C the *formal complement* determined by the Poincaré embedding. For example, if M^n and N^n are smooth compact manifolds, then a smooth embedding $g \colon M \to N$ avoiding ∂N gives rise to a codimension zero Poincaré embedding whose formal complement is the closure of $N \smallsetminus g(M)$ in N.

Slightly more generally, we will say that a Poincaré embedding f as above is *induced* by a smooth embedding $g \colon M \to N$ if $f|M = g$, and $f|C$ restricts to a simple homotopy equivalence (of special triads) from C to the closure of $N \smallsetminus g(M)$ in N.

2.1.1. Theorem. *Let M^n and N^n be smooth compact, $n \geq 5$. Let $f \colon M \amalg_{\partial M} C \to N$ be a codimension zero Poincaré embedding (in shorthand notation). Let $\iota \colon \nu_M \to f^*\nu_N|M$ be a stable vector bundle isomorphism refining the canonical stable fiber homotopy equivalence determined by the codimension zero Poincaré embedding (see explanations below). Assume that f induces an isomorphism $\pi_1 C \to \pi_1 N$. Then, up to a homotopy, the pair (f, ι) is induced by a smooth embedding $g \colon M \to N$ avoiding ∂N.*

Explanations. By the characterization of Spivak normal fibrations, the codimension zero embedding determines a stable fiber homotopy equivalence from ν_M (viewed as a spherical fibration) to $f^*\nu_n|M$ (ditto). The stable vector bundle isomorphism ι also determines such a stable fiber homotopy equivalence; we want the two to be fiberwise homotopic.

There is a mild generalization of 2.1.1 which involves the concept of a Poincaré embedding of arbitrary (formal) codimension. Assume this time that $(M, \partial M)$ and $(N, \partial N)$ are (simple) Poincaré pairs, of formal dimensions m and n, where $n - m =: q \geq 0$. A Poincaré embedding of $(M, \partial M)$ in $(N, \partial N)$ consists of

- a fibration $E \to M$ with fibers homotopy equivalent to $\mathbb{S}^{q-1}$ (the *unstable normal fibration* of the Poincaré embedding)
- a codimension zero Poincaré embedding of $(zE, \partial zE)$, where zE is the mapping cylinder of $E \to M$ and ∂zE is the union of E and the portion of zE projecting to ∂M.

This concept is due to [Br2], at least in the case where M and N are smooth manifolds, $\partial M = \emptyset = \partial N$.

2.1.2. Corollary. *Let M^m and N^n be smooth compact, where $n \geq 5$, and $q := n - m$. Let a Poincaré embedding f of $(M, \partial M)$ in $(N, \partial N)$ be given, with formal complement $(C, \partial C)$; let a reduction of the structure 'group' $G(q)$ of its unstable normal fibration to $O(q)$ be given, refining the canonical reduction for the stable normal fibration. Suppose that the induced homomorphism $\pi_1 C \to \pi_1 N$ is an isomorphism. Then there exists a smooth embedding $M \to N \smallsetminus \partial N$ inducing (up to a homotopy) the given Poincaré embedding and the unstable refinement of the canonical reduction for the stable normal fibration.*

Explanations. Let f_M be the restriction of f to M. The unstable refinement of the canonical reduction etc. is a point in the homotopy fiber of an evident map

$$B\mathrm{O}(q)^M \quad \longrightarrow \quad \mathrm{holim}\,[B\mathrm{G}(q)^M \to B\mathrm{G}^M \leftarrow B\mathrm{O}^M\,]$$

over the point determined by the unstable normal fibration on M, the stable normal (vector) bundle $\nu_M - {f_M}^* \nu_N$ on M, and the stable spherical fibration determined by the stable normal vector bundle.

Browder came close to 2.1.1 in [Br1] and proved in [Br2] the special case of 2.1.2 where M and N are simply connected, $\partial M = \emptyset = \partial N$, and $n - m \geq 3$, which makes the hypothesis on fundamental groups superfluous. One understands that Casson and Sullivan in unpublished but possibly mimeographed work and lectures simplified Browder's proof and obtained the appropriate uniqueness statement (see 2.2). Also, Casson pointed out [Ca] that Browder's hypothesis $n - m \geq 3$ could be replaced by the hypothesis on fundamental groups in 2.1.2. Wall [Wa2, ch.11] proved 2.1.2 in the nonsimply connected case. Therefore 2.1.2 and variations, see 2.2, are known as the Browder–Casson–Sullivan–Wall theorem. For an indication of the proof, see also 2.2.

2.2. Smoothing block families of Poincaré embeddings

Assume that M and N are smooth closed, for simplicity. The smooth embedding $M \to N$ whose existence is asserted in 2.1.2 is not determined up to isotopy, in general. But a relative version of 2.1.2, see [Wa2, 11.3 rel], implies that it is determined up to a concordance of embeddings (smooth embedding $M \times [0,1] \to N \times [0,1]$ taking $M \times i$ to $N \times i$ for $i = 0, 1$). In this way, 2.1.2 and the relative version give a homotopy theoretic expression for $\pi_0 \operatorname{emb}(M,N)$ modulo the concordance relation.

The block embedding space $\operatorname{emb}^{\sim}(M,N)$ is a crude approximation from the right to $\operatorname{emb}(M,N)$. It is the geometric realization of an incomplete simplicial set (alias simplicial set without degeneracy operators, alias Δ–set) whose k–simplices are the smooth embeddings of special manifold $(k+2)$–ads

$$\Delta^k \times M \longrightarrow \Delta^k \times N\,.$$

It is fibrant (has the Kan extension property), so that $\pi_k \operatorname{emb}^{\sim}(M,N)$, with respect to a base vertex $f\colon M \to N$, can be identified with the set of concordance classes of embeddings $\Delta^k \times M \to \Delta^k \times N$ which agree with $\mathrm{id} \times f$ on $\partial\Delta^k \times M$. Therefore 2.1.2 and the relative version give a homotopy theoretic expression for all $\pi_k \operatorname{emb}^{\sim}(M,N)$, $k \geq 0$. This suggests that 2.1.2 plus relative version admits a space level reformulation, involving $\operatorname{emb}^{\sim}(M,N)$ and a Poincaré embedding analogue. We denote that analogue by $\operatorname{emb_{PD}}^{\sim}(M,N)$; it is defined whenever M and N are Poincaré spaces. (There is also an 'unblocked' version, $\operatorname{emb_{PD}}(M,N)$; but the inclusion of $\operatorname{emb_{PD}}(M,N)$ in $\operatorname{emb_{PD}}^{\sim}(M,N)$ is a homotopy equivalence.)

We will also need notation and terminology for the complicated normal bundle and normal fibration data. Given Poincaré spaces M and N, of formal dimensions m and n, where $n-m =: q \geq 0$, a *Poincaré immersion* from M to N is a triple (f,ξ,ι) where $f\colon M \to N$ is a map, ξ^q is a spherical fibration on M (fibers $\simeq \mathbb{S}^{q-1}$), and ι is a stable fiber homotopy equivalence of the Spivak normal fibration ν_M with the Whitney sum alias fiberwise join $\xi \oplus f^*\nu_N$. We can make a space $\operatorname{imm_{PD}}(M,N)$ out of such triples; we can also use the $(k+2)$–ad analogue of the notion of Poincaré immersion to define a block immersion space $\operatorname{imm_{PD}}^{\sim}(M,N)$. It is easy to see that the inclusion of $\operatorname{imm_{PD}}(M,N)$ in $\operatorname{imm_{PD}}^{\sim}(M,N)$ is a homotopy equivalence.

Remarks. Suppose that M^m and N^n are smooth and closed, $n > m$. The immersion classification theorem, applied craftily to spaces of (smooth) block immersions, implies that the block immersion space $\operatorname{imm}^{\sim}(M,N)$ maps by a homotopy equivalence to the space of triples (f,ξ,ι) where

- $f\colon M \to N$ is a map
- ξ is an $(n-m)$–dimensional vector bundle on M
- $\iota\colon \nu_M \cong \xi \oplus f^*\nu_N$ is a stable vector bundle isomorphism.

This motivates our definition of $\operatorname{imm_{PD}}^{\sim}(M,N)$ for Poincaré spaces M and N, which is taken from [Kln3]. Beware: in the smooth setting, the inclusion $\mathrm{imm}(M,N) \to \mathrm{imm}^{\sim}(M,N)$ is not a homotopy equivalence in general. (Try $M = *$ and $N = \mathbb{S}^n$.)

We will sometimes also speak of Poincaré immersions from a Poincaré pair $(M, \partial M)$ to a Poincaré pair $(N, \partial N)$, of formal dimensions m and n, respectively, $m \le n$. The definition is much the same as before.

2.2.1. Theorem (Browder–Casson–Sullivan–Wall)**.** *For closed smooth M^m and N^n with $n \ge 5$ and $n - m \ge 3$, the following commutative square is ∞–cartesian:*

$$\begin{array}{ccc}
\mathrm{emb}^{\sim}(M,N) & \xrightarrow{\subset} & \mathrm{imm}^{\sim}(M,N) \\
\downarrow & & \downarrow \\
\mathrm{emb_{PD}}^{\sim}(M,N) & \xrightarrow{\subset} & \mathrm{imm_{PD}}^{\sim}(M,N)\,.
\end{array}$$

Remarks. The vertical arrows are essentially forgetful, but to make the one on the left hand side explicit, we ought to redefine $\mathrm{emb}^{\sim}(M,N)$ using smooth embeddings with specified riemannian tubular neighborhoods. The right hand vertical arrow is $(2n-3-3m)$–connected; therefore so is the left hand one. See [Wa2, Cor. 11.3.2].

If $n < 5$ we can still say that the square becomes ∞–cartesian when Ω^{5-n} is applied — this requires a choice of base vertex in $\mathrm{emb}^{\sim}(M,N)$. However, some condition like $n - m \ge 3$ is essential.

Theorem 2.2.1 has PL and TOP versions. In the PL and TOP settings, the content of the theorem is quite simply that $\mathrm{emb}^{\sim}(M,N)$ maps by a homotopy equivalence to $\mathrm{emb_{PD}}^{\sim}(M,N)$. Namely, in the PL and TOP settings, the right hand vertical arrow in the diagram in 2.2.1 is a homotopy equivalence; here again $n - m \ge 3$ is essential. See [Wa2, Cor. 11.3.1].

Example. We calculate $\mathrm{emb}^{\sim}(*, \mathbb{R}^n)$. Observe first that our definition of $\mathrm{emb}^{\sim}(M,N)$ makes sense for arbitrary smooth M and N without boundaries. We will use the stronger version of 2.2.1 where M and N are allowed to be compact with boundary; only smooth and Poincaré embeddings avoiding ∂N are considered, and we denote the corresponding block embedding spaces by $\mathrm{emb}^{\sim}(M,N)$ and $\mathrm{emb}^{\sim}_{\mathrm{PD}}(M,N)$ for brevity. Our choice is $M = *$ and $N = \mathbb{D}^n$ and we find

$$\begin{aligned}
\mathrm{imm}^{\sim}(*, \mathbb{D}^n) &\simeq \mathrm{O}/\mathrm{O}(n)\,, \\
\mathrm{imm_{PD}}^{\sim}(*, \mathbb{D}^n) &\simeq \mathrm{G}/\mathrm{G}(n)\,, \\
\mathrm{emb_{PD}}^{\sim}(*, \mathbb{D}^n) &\simeq *\,.
\end{aligned}$$

Therefore $\mathrm{emb}^{\sim}(*, \mathbb{R}^n) \cong \mathrm{emb}^{\sim}(*, \mathbb{D}^n)$ is homotopy equivalent to the homotopy fiber of the inclusion $\mathrm{O}/\mathrm{O}(n) \longrightarrow \mathrm{G}/\mathrm{G}(n)$.

2.2.2. Theorem. *Let M^n and N^n be smooth compact, $n \geq 5$. Assume that M has a handle decomposition with handles of index $\leq n-3$ only. Then the following commutative square is ∞–cartesian:*

$$\begin{array}{ccc} \mathrm{emb}^{\sim}(M,N) & \xrightarrow{\subset} & \mathrm{imm}^{\sim}(M,N) \\ \big\downarrow & & \big\downarrow \\ \mathrm{emb_{PD}}^{\sim}(M,N) & \xrightarrow{\subset} & \mathrm{imm_{PD}}^{\sim}(M,N)\,. \end{array}$$

Remarks. This is the 'family' version of 2.1.1. In particular, $\mathrm{emb}^{\sim}(M,N)$ is short for the space of smooth block embeddings of M in $N \smallsetminus \partial N$, and so on. Precise definitions of the spaces in the diagram are left to the reader. It is easy to deduce 2.2.1 from 2.2.2. We could drop the condition $n \geq 5$ at the price of choosing a base vertex in $\mathrm{emb}^{\sim}(M,N)$ and applying Ω^{5-n}.

We turn to the proof of 2.2.2, assuming $\partial N = \emptyset$ for simplicity. Actually we will just deduce 2.2.2 from the Sullivan–Wall(–Quinn–Ranicki) homotopy fiber sequence. To explain what this is, we fix a (simple) Poincaré space W of formal dimension n. An *s–structure* on W is a pair (M,f), where M is smooth closed and f is a simple homotopy equivalence $f\colon M \to W$. The s–structures on W form a groupoid where an isomorphism from (M_1,f_1) to (M_2,f_2) is a diffeomorphism $g\colon M_1 \to M_2$ satisfying $f_2 g = f_1$. We enlarge this to an incomplete simplicial groupoid $\mathrm{str}_\bullet(W)$ in which $\mathrm{str}_k(W)$ is the groupoid of s–structures, in the special $(k+2)$–ad sense, on $\Delta^k \times W$. The *block structure space* $\mathcal{S}^{\sim}(W)$ of W can be defined as the geometric realization of the incomplete simplicial set whose simplices in degree k are rules ρ which associate

- to each face z of Δ^k, an object $\rho(z)$ in $\mathrm{str}_{|z|}(W)$;
- to each face z of Δ^k and face operator δ from degree $|z|$ to a smaller degree, an isomorphism $\delta\rho(z) \to \rho\delta(z)$ in $\mathrm{str}_{|z|}(W)$. (These isomorphisms are subject to an evident associativity condition.)

There is a second definition of $\mathcal{S}^{\sim}(W)$, homotopy equivalent to the first, according to which $\mathcal{S}^{\sim}(W)$ is the geometric realization of the incomplete simplicial space $k \mapsto |\mathrm{str}_k(W)|$. However, the first definition matches our earlier definition of block embedding spaces better. — The Sullivan–Wall homotopy fiber sequence is then

$$\mathcal{S}^{\sim}(W) \longrightarrow R^{\mathrm{O}}_{\mathrm{G}}(\nu_W) \longrightarrow \Omega^{n+\infty} \boldsymbol{L}^s_\bullet(\mathbb{Z}\pi_1 W).$$

Here $R^{\mathrm{O}}_{\mathrm{G}}(\nu_W)$ is the homotopy fiber of $B\mathrm{O}^W \to B\mathrm{G}^W$ over the point determined by ν_W (think of it as the space of 'reductions' of the structure 'group' of ν_W, from G to O) and $\mathbf{L}^s_\bullet(\mathbb{Z}\pi_1 W)$ is the L–theory spectrum associated with the group ring $\mathbb{Z}\pi_1 W$. We have shortened $\Omega^n\Omega^\infty$ to $\Omega^{n+\infty}$. We need a slightly more complicated version where W is a Poincaré triad, $\partial W = \partial_0 W \cup \partial_1 W$, and the s–structures are fixed (prescribed) on $\partial_0 W$. Consequently the structure 'group' reductions are also fixed (prescribed) over $\partial_0 W$, and the relevant L–theory spectrum is the one associated with the homomorphism of group rings $\mathbb{Z}\pi_1\partial_1 W \to \mathbb{Z}\pi_1 W$ induced by inclusion.

With M and N as in 2.2.2, fix a Poincaré embedding of M in N, say $f\colon M \amalg_{\partial M} C \to N$. Let W be the mapping cylinder of f. Then W is a Poincaré triad with $\partial W \cong (M \amalg_{\partial M} C) \amalg N$ and $\partial_0 W = M \amalg N$, $\partial_1 W = C$. There is a preferred s–structure on $\partial_0 W$, given by the identity; indeed $\partial_0 W$ is a smooth compact manifold. Browder's crucial, highly original and yet trivial observation at this point, slightly reformulated, is that $\mathcal{S}^\sim(W)$, with the definition where structures are prescribed on $\partial_0 W$, is homotopy equivalent to the homotopy fiber (over the point f) of the left hand vertical arrow in the diagram of 2.2.2. It is easy to check that the corresponding homotopy fiber of the right hand vertical arrow is homotopy equivalent to $R^{\mathrm{O}}_{\mathrm{G}}(\nu_W)$ (with the definition where the reductions are fixed over $\partial_0 W$), and that, with these identifications, the map of homotopy fibers becomes the map

$$\mathcal{S}^\sim(W) \longrightarrow R^{\mathrm{O}}_{\mathrm{G}}(\nu_W)$$

from the Sullivan–Wall homotopy fiber sequence. It is a homotopy equivalence because, by the π–π–theorem, the L–theory term in the Sullivan–Wall homotopy fiber sequence is contractible. □

Remark. In this proof $R^{\mathrm{O}}_{\mathrm{G}}(\nu_W)$ can be interpreted, via transversality, as a space of 'degree one normal maps' to W which restrict to identity maps over $\partial_0 W$. Such a normal map to W is exactly the same thing as a smooth embedding $M \to N'$, plus a degree one normal map $N' \to N$, plus a normal cobordism from $N' \to N$ to the identity $N \to N$.

2.3. Embedded Surgery

Let M^m be smooth closed. Following Levine [Lev] and Rigdon–Williams [RiW], we will discuss the construction of embeddings $M \to \mathbb{R}^n$ from the following data:

- a degree one normal map $g\colon M' \to M$ and a normal (co)bordism h from g to id: $M \to M$;
- a smooth embedding $e\colon M' \to \mathbb{R}^n$.

On the set of such triples (g, h, e) there is an evident bordism relation. Surgery methods can be used (some details below) to show that, when $2n-3-3m \geq 0$, every bordism class has a representative (g, h, e) in which h is a product cobordism, so that g is homotopic to a diffeomorphism. A straightforward Thom–Pontryagin construction leads to a homotopy theoretic description of the bordism set. Combining these two ideas, one obtains embeddings $M \to \mathbb{R}^n$ from homotopy theoretic data if $2n-3-3m \geq 0$.

The homotopy theoretic description. Let $\nu = \nu_M$ be the stable normal bundle of M. Let $V^{n-m}(\nu)$ be the tautological $(n-m)$–dimensional vector bundle on the homotopy pullback of

$$M \xrightarrow{\nu} B\mathrm{O} \leftarrow B\mathrm{O}(n-m)\,.$$

There is a forgetful map from the base of $V^{n-m}(\nu)$ to M, and a stable map of vector bundles $V^{n-m}(\nu) \to \nu$ covering it. This leads to another forgetful or stabilization map

$$\Omega^n T(V^{n-m}(\nu)) \to \Omega^{m+\infty}\boldsymbol{T}(\nu) \tag{2.3.1}$$

where the T denotes a Thom space and the (boldface) $\boldsymbol{T}$ denotes a Thom spectrum. In $\Omega^{m+\infty}\boldsymbol{T}(\nu)$ we have a distinguished point ρ, the Spanier–Whitehead or Poincaré dual of $1\colon M \to Q\mathbb{S}^0$. See 0.2, on the subject of Poincaré spaces. By a Thom–Pontryagin construction, the set of triples (g, h, e) as above, modulo bordism, can be identified with π_0 of the homotopy fiber of (2.3.1) over the point ρ.

Digressing a little now, we note that a smooth embedding $M \to \mathbb{R}^n$ determines a triple (g, h, e) as above where $h\colon M \times [0,1] \to M$ is the projection and e from $M \times 1 \cong M$ to $\mathbb{R}^n$ is that embedding. The bordism class of the triple (g, h, e) may be called the (smooth, unstable, etc.) *normal invariant* of the embedding $M \to \mathbb{R}^n$.

The surgery methods. Assume that $m \geq 5$. Let (g, h, e) be one of those triples. To be more specific, we write the normal cobordism in the form $h\colon M'' \to M$, where $\dim(M'') = m+1$. Surgery below the middle dimension on M'' creates a bordism from the triple (g, h, e) to another such triple, (g_1, h_1, e_1), in which $h_1\colon M_1'' \to M$ is $[m/2+1/2]$–connected. Then the inclusion of M in M_1'' is $[m/2-1/2]$–connected. It follows that M_1'' admits a handle decomposition, relative to a collar on $M_1' := \partial M_1'' \smallsetminus M$, with handles of index $\leq m - [m/2-1/2]$ only.

Now choose a handle of lowest index i, giving a framed embedding $u\colon (\mathbb{D}^i, \mathbb{S}^{i-1}) \to (M_1'', M_1')$. We try to create a bordism from (g_1, h_1, e_1) to another triple, (g_2, h_2, e_2), by doing a half–surgery, alias handle subtraction, on $u(\mathbb{D}^i)$. Of course, the resulting surgery on $u(\mathbb{S}^{i-1}) \subset M_1'$ has to be an embedded surgery — embedded in $\mathbb{R}^n \times [0,1]$, to be precise. The 'embedded surgery lemma' in [RiW] shows that the required (partly embedded) half–surgery can be carried out if $n-m > i$. Since $i \le m-[m/2-1/2]$, it is enough to require $2n-3-3m \ge 0$. In that case we can also repeat the procedure until all handles in the handle decomposition of M_1'' relative to M_1' have been subtracted. So (g_2, h_2, e_2) is bordant to a triple (g_r, h_r, e_r) in which h_r is a product cobordism. □

Hence we have the existence part of the following (the proof of uniqueness uses the same ideas in a relative setting):

2.3.2. Proposition [RiW]. *Assume $m \ge 5$. If $2n-3-3m \ge 0$, then every element in π_0 of the homotopy fiber of* (2.3.1) *over ρ is the (unstable) normal invariant of a smooth embedding $M \to \mathbb{R}^n$. If $2n-3-3m > 0$, such an embedding is unique up to concordance.*

Although 2.3.2 owes a lot to the ideas in [Lev], it has a sharper focus and leads on to a number of new ideas. In particular, 2.3.2 generalizes easily to block families: M can be replaced by $M \times \Delta^k$ and $\mathbb{R}^n$ by $\mathbb{R}^n \times \Delta^k$ in the sketch proof. We must require $2(n+k)-3-3(m+k) \ge 0$, in other words $k \le 2n-3-3m$, and pay some attention to the faces $M \times d_i\Delta^k$. This shows that our unstable normal invariant for embeddings of M in $\mathbb{R}^n$ is really a map

$$\begin{array}{c} \mathrm{emb}^\sim(M,\mathbb{R}^n) \\ \big\downarrow \\ \mathrm{hofiber}_\rho\,[\,\Omega^n T(V^{n-m}(\nu)) \to \Omega^{m+\infty}\boldsymbol{T}(\nu)\,] \end{array} \qquad (2.3.3)$$

and gives us an estimate for the connectivity:

2.3.4. Theorem. *The map* (2.3.3) *is* $(2n-3-3m)$*–connected* ($m \ge 5$).

Let $f\colon M \to \mathbb{R}^n$ be an immersion with normal bundle ν_f; so ν_f is a vector bundle of dimension $n-m$ on M.

2.3.5. Corollary. *Suppose that $m \ge 5$. There is a* $(2n-3-3m)$*–connected map*

$$\begin{array}{c} \mathrm{hofiber}_f\,[\,\mathrm{emb}^\sim(M,\mathbb{R}^n) \hookrightarrow \mathrm{imm}^\sim(M,\mathbb{R}^n)\,] \\ \big\downarrow \\ \mathrm{hofiber}_\rho\,[\,\Omega^n T(\nu_f) \hookrightarrow \Omega^{m+\infty}\boldsymbol{T}(\nu)\,]\,. \end{array}$$

Proof. The map is a variation on (2.3.3); use the fact that an embedding $M \to \mathbb{R}^n$ equipped with a regular homotopy to f has a normal bundle which is canonically identified with ν_f. To show that the map in question is $(2n-3-3m)$–connected, view it as the left column of a commutative square whose right column is (2.3.3). Now we need to show that the square is $(2n-3-3m)$–cartesian. With the abbreviations $\mathrm{emb}^\sim = \mathrm{emb}^\sim(M, \mathbb{R}^n)$ and $\mathrm{imm}^\sim = \mathrm{imm}(M, \mathbb{R}^n)$, this reduces to the assertion that

$$\begin{tikzcd} \mathrm{hofiber}_f[\mathrm{emb}^\sim \to \mathrm{imm}^\sim] \arrow[r, "\text{forget}"] \arrow[d] & \mathrm{emb}^\sim \arrow[d] \\ \Omega^n T(\nu_f) \arrow[r] & \Omega^n T(V^{n-m}(\nu)) \end{tikzcd}$$

is $(2n-3-3m)$–cartesian. Actually this is $(2n-2-3m)$–cartesian. (Use the immersion classification theorem and Blakers–Massey to understand the homotopy fibers of the upper and lower rows, respectively. Then compare.) $\square$

2.4. Poincaré embeddings into disks

Williams was apparently the first to realize [Wi1] that the proper context for Levine's ideas in [Lev] was not manifold geometry, but Poincaré space geometry. To illustrate this point, we will translate 2.3.5 into Poincaré space language, relying on 2.2.1 for the translation. For a Poincaré pair $(M, \partial M)$ of formal dimension n, let $\Omega^n_\vdash(M/\partial M) \subset \Omega^n(M/\partial M)$ consist of the elements which carry a fundamental class for $(M, \partial M)$. This is a union of connected components of $\Omega^n(M/\partial M)$.

2.4.1. Reformulation of 2.3.5. *Let $(M, \partial M)$ be the Poincaré pair of formal dimension n determined by a spherical fibration $\xi^{n-\ell}$ on a smooth closed L^ℓ. That is, ∂M is the total space of ξ, and M is the mapping cylinder of the projection $\partial M \to L$. If $\ell \geq 5$, then the map*

$$\mathrm{emb}^\sim_{\mathrm{PD}}(M, \mathbb{D}^n) \longrightarrow \Omega^n_\vdash(M/\partial M)$$

associating to a Poincaré embedding its collapse map is $(2n-3-3\ell)$–connected.

Explanation. Make a space E_{PD} whose elements are pairs (μ, σ) where $\mu^{n-\ell}$ is a spherical fibration on L, and $\sigma\colon \mathbb{S}^n \to T(\mu)$ carries a fundamental class. The real content of 2.4.1 is that the map

$$\mathrm{emb}^\sim_{\mathrm{PD}}(L, \mathbb{D}^n) \to E_{\mathrm{PD}}$$

which to a Poincaré embedding associates its normal bundle and collapse map is $(2n-3-3\ell)$–connected. We now deduce this from 2.3.5:

By the characterization of the Spivak normal fibration of L, the spherical fibration μ in any $(\mu,\sigma)\in E_{\mathrm{PD}}$ is (canonically) stably fiber homotopy equivalent to ν_L. So there is a forgetful map $E_{\mathrm{PD}}\to \mathrm{imm}^{\sim}_{\mathrm{PD}}(L,\mathbb{D}^n)$. Let E be the homotopy pullback of

$$E_{\mathrm{PD}}\to \mathrm{imm}^{\sim}_{\mathrm{PD}}(L,\mathbb{D}^n)\leftarrow \mathrm{imm}^{\sim}(L,\mathbb{R}^n)\,.$$

We get a commutative square

$$\begin{array}{ccc}
\mathrm{emb}^{\sim}(L,\mathbb{R}^n) & \longrightarrow & E \\
\big\downarrow & & \big\downarrow \\
\mathrm{emb}^{\sim}_{\mathrm{PD}}(L,\mathbb{D}^n) & \longrightarrow & E_{\mathrm{PD}}
\end{array}$$

which is ∞–cartesian by 2.2.1. By the remark after 2.2.1, the right–hand vertical arrow is $(2n-3-3\ell)$–connected So it is enough to show that the upper horizontal arrow in the square is $(2n-3-3\ell)$–connected. But that is exactly the content of 2.3.5. $\square$

Williams saw that the peculiar hypotheses on the Poincaré pair $(M/\partial M)$ in 2.4.1 could be replaced by a single much simpler one. (For simplicity we restrict attention to $\pi_0\,\mathrm{emb}^{\sim}_{\mathrm{PD}}(M,\mathbb{D}^n)$. We write $\pi_n^{\vdash}(M/\partial M)$ for the subset of $\pi_n(M/\partial M)$ consisting of the elements which carry a fundamental class.)

2.4.2. Theorem [Wi1]**.** *Let $(M,\partial M)$ be a Poincaré pair of formal dimension $n\geq 6$, where M is homotopy equivalent to a CW–space of dimension m. Assume that $\pi_1\partial M\to\pi_1 M$ is an isomorphism. Then the map*

$$\pi_0\,\mathrm{emb}^{\sim}_{\mathrm{PD}}(M,\mathbb{D}^n)\longrightarrow \pi_n^{\vdash}(M/\partial M)$$

associating to a Poincaré embedding the class of its collapse map is surjective for $2n-3-3m\geq 0$, and bijective for $2n-3-3m>0$.

Williams' proof of 2.4.2 uses Hodgson's thickening theorem, 2.4.4 below. This is a distant corollary of Hudson's embedding theorem:

2.4.3. Theorem [Hu1, 8.2.1], [Hu2, 1.1]**.** *If N^n is a compact smooth manifold and P is a codimension zero compact smooth submanifold of ∂N such that $P\hookrightarrow N$ is j–connected, then any element of $\pi_r(N,P)$ may be represented by a smooth embedding $(\mathbb{D}^r,\mathbb{S}^{r-1})\to(N,P)$ provided that $r\leq n-3$ and $2r\leq n+j-1$.*

2.4.4. Theorem [Ho, 2.3]. *Let N^n be a compact smooth manifold, $n \geq 6$, and P a codimension zero smooth submanifold of ∂N. Let K be a CW–space rel P of (relative) dimension $\leq k$, and let $f\colon K \to N$ be any map rel P. If f is $(2k-n+1)$–connected rel P, then f is homotopic rel P to a composition*

$$K \xrightarrow{\simeq} K' \hookrightarrow N$$

where K' is a smooth compact triad contained in N with $\partial_0 K' = P = K' \cap \partial N$, and $K \to K'$ is a simple homotopy equivalence rel P.

Remark. It is an exercise, but a non–trivial one, to deduce the special case of 2.4.4 where K has just one cell rel P from 2.4.3.

Outline of proof of 2.4.2. Two key concepts in Williams' proof are those of compression and decompression. The *decompression* of a codimension zero Poincaré embedding of M in N is an obvious codimension zero Poincaré embedding of $M \times J$ in $N \times I$ where $I = [0,1]$ and $J = [1/3, 2/3]$. Here M, N are short for Poincaré pairs of formal dimension n, and $M \times J$, $N \times I$ are short for certain Poincaré pairs of formal dimension $n+1$. Conversely, to *compress* a Poincaré embedding of $M \times J$ in $N \times I$ means to find a concordance from it to the decompression of some Poincaré embedding of M in N.

Browder points out in [Br2] that a map $\eta\colon \mathbb{S}^n \to M/\partial M$ which carries a fundamental class for the Poincaré pair $(M, \partial M)$ determines a Poincaré embedding of $M \times J$ in $\mathbb{D}^n \times I$. Its formal complement C is the mapping cylinder of

$$\partial(M \times J) \amalg \partial(\mathbb{D}^n \times I) \quad \xrightarrow{q \amalg \eta} \quad M/\partial M$$

where q is the quotient map collapsing $M \times 1/3$ and $\partial M \times J$ to a point. The boundary ∂C is $\partial(M \times J) \amalg \partial(\mathbb{D}^n \times I)$.

This leaves the task of compressing $M \times J \to \mathbb{D}^n \times I$, the Poincaré embedding determined by some η from $\mathbb{S}^n$ to $M/\partial M$ as above, to a Poincaré embedding $M \to \mathbb{D}^n$. Hirsch [Hi2] gives a necessary and often sufficient condition for the existence of such a compression: that the inclusion of $M \times 1/3 \subset \partial(M \times J)$ in the formal complement C of the Poincaré embedding $M \times J \to \mathbb{D}^n \times I$ be nullhomotopic. This is clearly satisfied here — there is a preferred choice of nullhomotopy alias *link trivialization.* Williams shows in fact that the map just described, from $\pi_n^{\vdash}(M/\partial M)$ to concordance classes of Poincaré embeddings $M \times J \to \mathbb{D}^n \times I$ with a link trivialization, is a bijection. (This is not difficult.) He then proceeds to show that the link trivialization determines a compression. His argument has two parts:

(i) Without loss of generality, M and C are compact smooth manifolds. Namely, the existence of $\eta\colon \mathbb{S}^n \to M/\partial M$ implies that the Spivak normal fibration of M is trivial; hence there exists a degree one normal map $(M,\partial M) \to (M,\partial M)$ where M is smooth compact, and by the $\pi-\pi$ theorem (here we use $n \geq 6$ and the condition on fundamental groups) this is normal bordant to a homotopy equivalence. A similar argument works for C ; in this case the manifold structure is already prescribed on $\partial_0 C$ since we want $\partial_0 C \cong \partial M$.

(ii) The nullhomotopy for $M \times 1/3 \hookrightarrow C$ means that the inclusion of $(M \times 1/3) \amalg (\mathbb{D}^n \times 0)$ in ∂C extends to a map $e\colon X \to C$, where X is any CW–space rel $(M \times 1/3) \amalg (\mathbb{D}^n \times 0)$ which is contractible. The metastable range condition in 2.4.2 now makes it possible to use Hodgson's thickening theorem, 2.4.4. The conclusion is that X can be taken to be a compact $n+1$–manifold with $\pi_1 \partial X \cong \pi_1 X$, and $(M \times 1/3) \amalg (\mathbb{D}^n \times 0)$ contained in ∂X ; moreover, e can be taken to be an embedding. Then X is an $(n+1)$–disk and the inclusion of $M \times 1/3$ in the closure of $\partial X \smallsetminus (\mathbb{D}^n \times 0)$ is the compressed embedding we have been looking for. $\square$

Remark. The idea to use Hodgson's thickening theorem 2.4.4 for compression purposes comes from [Lt] and Hirsch [Hi2]. Actually Hirsch had to work with Hudson's embedding theorem, 2.4.3.

Williams noted in [Wi2] that his own proof of 2.4.2 "... consists of converting $(M,\partial M)$ to a manifold and then using smooth embedding theory " and went on to propose an alternative and truly homotopy theoretic proof, along the following lines. Given $\eta\colon \mathbb{S}^m \to M/\partial M$ carrying a fundamental class, Browder's observation gives as before a Poincaré embedding of $M \times J$ in $\mathbb{D}^n \times I$ with a preferred link trivialization, and formal complement homotopy equivalent to $M/\partial M$. If this compresses to a Poincaré embedding of M in $\mathbb{D}^n$ with formal complement A, then there is a square, commutative up to preferred homotopy

$$\begin{tikzcd} \partial(M \times J) \arrow[r] \arrow[d, "\text{quotient map}"'] & M/\partial M \arrow[d, "\simeq"] \\ \Sigma_u \partial M \arrow[r, "\Sigma_u \iota"] & \Sigma_u A. \end{tikzcd}$$

Here Σ_u denotes unreduced suspension, and the rows are, respectively and essentially, inclusion of boundary of $M \times J$ in complement of uncompressed embedding, and Σ_u of inclusion of boundary of M in complement of compressed embedding. The left–hand column is (isomorphic to) the projection from $\partial(M \times J)$ to the quotient of $\partial(M \times J)$ by $M \times \partial J$. — Conversely, if the square exists, then the compression exists. Using elementary homotopy theoretic arguments, Williams managed to show that, under the hypotheses of 2.4.2, there is indeed a homotopy commutative square

$$\begin{array}{ccc} \partial(M \times J) & \longrightarrow & M/\partial M \\ {\scriptstyle \text{quotient map}}\Big\downarrow & & \Big\downarrow{\scriptstyle \simeq} \\ \Sigma_u \partial M & \longrightarrow & \Sigma_u A. \end{array}$$

But he did not show with homotopy theoretic methods that the lower horizontal arrow desuspends. This was done much later by Richter [Ric], who combined desuspension techniques of Hilton and Boardman–Steer [BS], Berstein–Hilton [BH], and Ganea [Ga1], [Ga2], [Ga3].

2.5. Poincaré embeddings: The fiberwise point of view

We turn to the subject of codimension zero Poincaré embeddings with arbitrary codomain. To remain as close as possible to the conceptual framework of 2.4, we use the language and notation of fiberwise homotopy theory (over the codomain, which is fixed). The idea of using fiberwise homotopy theory in the context of Poincaré duality and Poincaré embeddings is due to J. Klein and S. Weinberger, independently.

Notation, terminology. For a space Z, let $\mathcal{R}(Z)$ be the category of retractive spaces over Z. An object of $\mathcal{R}(Z)$ is a space C equipped with maps $r\colon C \to Z$ and $s\colon Z \to C$ such that $rs = \mathrm{id}_Z$. We assume that s is a cofibration. The morphisms from C_1 to C_2 are maps $f\colon C_1 \to C_2$ satisfying $fs_1 = s_2$ and $r_2 f = r_1$ where r_i and s_i are the structure maps for C_i. We call such a morphism a weak equivalence if the underlying map $C_1 \to C_2$ of spaces, without structure maps from and to Z, is a homotopy equivalence. (We will make sure that all spaces in sight are homotopy equivalent to CW–spaces.) If r_2 is a fibration, we define $[C_1, C_2]$ as the set of homotopy classes (vertical and rel Z) of morphisms $C_1 \to C_2$ in $\mathcal{R}(N)$. In general, we choose a weak equivalence $C_2 \to C_2^{\&}$ of retractive spaces over Z, where the structure map $C_2^{\&} \to Z$ is a fibration, and let

$$[C_1, C_2] := [C_1, C_2^{\&}]\,.$$

More notation. For a space X over Z and (well–behaved) subspace A of X, let $X/\!\!/A$ be the pushout of $X \leftarrow A \to Z$, viewed as an object of $\mathcal{R}(Z)$ with obvious structure maps.

Let $(M, \partial M)$ and $(N, \partial N)$ be Poincaré pairs of the same formal dimension n. Let $g\colon M \to N$ be any map (not necessarily respecting boundaries). If

$$f\colon (M \amalg_{\partial M} C\,,\, \partial_1 C) \longrightarrow (N, \partial N)$$

is any Poincaré embedding of M in N, then we can regard the domain of f as a space over N. If f is equipped with the additional structure of a homotopy from $f|M$ to g, then the identification map from $(M \amalg_{\partial M} C) /\!\!/ \partial_1 C$ to $(M \amalg_{\partial M} C) /\!\!/ C$ can be written, modulo canonical weak equivalences, as a map

$$\eta\colon N /\!\!/ \partial N \longrightarrow M /\!\!/ \partial M$$

where M is viewed as a space over N by means of g. We call η the collapse map determined by f (and the homotopy from $f|M$ to g). It carries a fundamental class for $(M, \partial M)$. That is, the induced map from $N/\partial N$ to $M/\partial M$ takes any fundamental class for $(N, \partial N)$ to one for $(M, \partial M)$. Let

$$[N /\!\!/ \partial N, M /\!\!/ \partial M]^{\vdash} \subset [N /\!\!/ \partial N, M /\!\!/ \partial M]$$

consist of the homotopy classes of retractive maps $N /\!\!/ \partial N \to (M /\!\!/ \partial M)^{\&}$ which are fundamental–class carrying.

2.5.1. Theorem [Kln3]**.** *Let $(M, \partial M)$ and $(N, \partial N)$ be Poincaré pairs of formal dimension n. Suppose that M has the homotopy type of a CW–space of dimension m, and $\partial M \to M$ induces an isomorphism of fundamental groups. Let $g\colon M \to N$ be any map. Then the map*

$$\pi_0\, \mathrm{hofiber}_g\, (\mathrm{emb}^{\sim}_{\mathrm{PD}}(M, N) \to \mathrm{map}(M, N)) \quad \longrightarrow \quad [N /\!\!/ \partial N, M /\!\!/ \partial M]^{\vdash}$$

which associates to a Poincaré embedding f (with a homotopy from $f|M$ to g) its collapse map is surjective for $2n - 4 - 3m \geq 0$, and bijective for $2n - 4 - 3m > 0$.

Outline of proof, following [Kln3]. The proof is very similar to that of 2.4.2. Make M into a space over N using g. Every $[\eta]$ in $[N /\!\!/ \partial N, M /\!\!/ \partial M]$ can be represented by a morphism

$$\eta\colon N /\!\!/ \partial N \longrightarrow (M /\!\!/ \partial M)^{\&}$$

in $\mathcal{R}(N)$. If η carries a fundamental class, then it determines a Poincaré embedding of $M \times J$ in $N \times I$ whose formal complement C is the mapping cylinder of

$$\partial(M \times J) \amalg \partial(N \times I) \quad \xrightarrow{q \amalg \eta} (M /\!\!/ \partial M)^{\&}.$$

The Poincaré embedding has a preferred link trivialization, a vertical null-homotopy (over N) of the composite map $M \times 1/3 \hookrightarrow \partial C \hookrightarrow C$. One shows that the link trivialization determines a compression. At this point, conversion of M and C to manifolds is not an option. So what one needs is an analogue of Hodgson's thickening theorem, 2.4.4, with Poincaré pairs instead of manifolds with boundary. Klein supplies this in [Kln1], [Kln2]. It is (currently) slightly less sharp than Hodgson's, which accounts for the loss of one dimension ($2n-4-3m$ in 2.5.1 where 2.4.2 has $2n-3-3m$). $\square$

Remarks. This proof is much closer to Williams' original proof of 2.4.2 than to the alternative homotopy theoretic proof of 2.4.2 planned by Williams and carried out by Richter.

Klein's proof of the Poincaré analogue of Hodgson's thickening theorem is homotopy theoretic, and it is the homotopy theory of retractive spaces over some fixed base space which is used.

3. Higher excision, multiple disjunction

Remark. The canonical problem in (approximate) higher excision theory of practically any kind is this. Given a finite set S and a strongly ∞–cocartesian S–cube X of spaces (perhaps subject to some conditions of a geometric nature), and a functor F, covariant or contravariant, from such spaces to spaces, find a large k such that the S–cube FX is k–cartesian. This was apparently first considered for $F = \mathrm{id}$ by Barratt and J.H.C. Whitehead [BaW], following the work of Blakers and Massey [BlM1], [BlM2] in the case $|S| = 2$. The result of [BaW] was later improved on by Ellis and Steiner [ES]; see also [Go3]. For us, X will often be a cube of manifolds, and F will often be something like 'space of embeddings to or from a fixed manifold'.

Notation, conventions. In this chapter, N denotes a compact smooth manifold with boundary, or a Poincaré pair of formal dimension n. The symbols M and L_i are reserved for (smooth compact or Poincaré) triads; here i runs through the elements of a finite set S. We assume that embeddings $\partial_0 M \to N$ and $\partial_0 L_i \to N$ are specified, with 'disjoint' images (in the Poincaré case this means that a Poincaré embedding of the disjoint union of M and the L_i in N is specified).

For $R \subset S$ write $L_R := \coprod_{i \in R} L_i$. In the smooth case, we allow only embeddings from M to N or from L_R to N which agree with the specified ones on $\partial_0 M$ or $\partial_0 L_R$, and for which $\partial_0 M$ or $\partial_0 L_R$ is the transverse preimage of ∂N. Analogous conditions are imposed in the Poincaré case ;

also, *maps* from M to N or from L_R to N are prescribed on $\partial_0 M$ and $\partial_0 L_R$. Spaces of such embeddings and maps will be denoted $\mathrm{emb}(M,N)$, $\mathrm{emb}(L_R\ ,N)$, $\mathrm{map}(M,N)$ and so on, with embellishments as appropriate, e.g., a tilde for block embedding spaces. If we wish to make the subscript R in L_R into a variable, we may write $L_\bullet$. For example, $\mathrm{emb}(L_\bullet, N)$ is short for the (contravariant) S–cube given by $R \mapsto \mathrm{emb}(L_R, N)$.

Sometimes, but not always, we assume $M \subset N$ and/or $L_i \subset N$, in which case the inclusions $M \to N$ and/or $L_i \to N$ are subject to the above conditions.

In the case where the L_i are smooth, let ℓ_i be the smallest number such that L_i can be obtained from a closed collar on $\partial_0 L_i$ by successively attaching handles of index $\leq \ell_i$. In the case where the L_i are Poincaré triads, let ℓ_i be the smallest number such that L_i is homotopy equivalent rel $\partial_0 L$ to a CW–space rel $\partial_0 L$ with cells of dimension $\leq \ell_i$ only. Let m be the corresponding number for M. (These numbers are called 'relative handle dimension' in the smooth case, and 'relative homotopy dimension' in the Poincaré case.) Let $\ell'_i := n - \ell_i - 2$.

3.1. Easy multiple disjunction for embeddings

Here we assume that M, N and L_i for $i \in S$ are smooth, and $L_S \subset N$.

3.1.1. Proposition. *The diagram* $\mathrm{emb}(M, N \smallsetminus L_\bullet)$ *is* $(1 + \Sigma_i(\lambda_i - 2))$–*cartesian, where* λ_i *is the maximum of* $(n - m - \ell_i)$ *and* 0.

Proof. Abbreviate $E_R = \mathrm{emb}(M, N \smallsetminus L_R)$ for $R \subset S$. By an easy multiple induction over the number(s) of handles needed to build M from $\partial_0 M$, and L_i from $\partial_0 L_i$, we can reduce to the case where these numbers are all equal to 1. We may then replace the handles by their cores; so now M and the L_i are disks of dimension m and ℓ_i , respectively, and $\partial_1 M = \emptyset$, $\partial_1 L_i = \emptyset$.

General position arguments show that the complement of $E_{R\cup i}$ in E_R, for $i \in S \smallsetminus R$, has codimension $\geq \lambda_i$ in E_R, and the complement of $\cup_{i \notin R} E_{R \cup i}$ has codimension $\geq \Sigma_{i \notin R} \lambda_i$ in E_R. Therefore each pair

$$(E_R, \cup_{i \notin R} E_{R \cup i})$$

is (k_{S-R})–connected where $k_T = -1 + \Sigma_{i \in T} \lambda_i$ for $T \subset S$. According to [Go3, 2.5] the cubical diagram is then k–cartesian where k is the minimum of $1 - |S| + \Sigma_\alpha k_{S(\alpha)}$ over all partitions $\{S(\alpha)\}$ of S. The minimum is achieved when S is partitioned into singletons. $\square$

In the corollary which follows, we have N and L_i for $i \in S$ as usual; there is no M and there is no preferred embedding of L_S in N.

3.1.2. Corollary. *The diagram* $\mathrm{emb}(L_\bullet, N)$ *is* $(3-n+\Sigma_i(n-2\ell_i-2))$–*cartesian.*

Proof. Choose $j \in S$ for which ℓ_j is minimal. Let $T := S \smallsetminus j$. It is enough to show that for every choice of base point e in $\mathrm{emb}(L_T, N)$, the cube given by

$$R \mapsto \mathrm{hofiber}_e \left[\mathrm{emb}(L_{R\cup j}, N) \xrightarrow{\text{res.}} \mathrm{emb}(L_R, N) \right]$$

for $R \subset T$ is $(3-n+\Sigma_{i\in S}(n-2\ell_i-2))$–cartesian. Here homotopy fibers over e may be replaced by fibers over e, so that we have to show that $R \mapsto \mathrm{emb}(L_j, N \smallsetminus e(L_R))$ is $(3-n+\Sigma_{i\in S}(n-2\ell_i-2))$–cartesian. But this follows directly from 3.1.1, with T instead of S and L_j instead of M. ☐

Remark/Preview. Although 3.1.2 is not sharp, it is an excellent tool in the study of spaces of smooth embeddings $\mathrm{emb}(M, N)$ when $2m < n-2$. To handle all cases $m < n-2$, we need a stronger multiple disjunction theorem for embeddings, 3.5.1 below. This is much harder to prove. We will proceed in historical order, going through multiple disjunction and higher excision theorems for spaces of concordance embeddings, Poincaré embeddings, and block embeddings, before we get to (serious) multiple disjunction and higher excision for spaces of embeddings.

3.2. Multiple disjunction for concordance embeddings

Here we assume that M, N and L_i for $i \in S$ are smooth, $M \subset N$ and $L_i \subset N$, pairwise disjoint in N. A *concordance embedding* of M in N is a concordance of embeddings from the inclusion to some other embedding, i.e. an embedding $M \times [0,1] \to N \times [0,1]$ which

- restricts to the inclusion on $M \times 0$ and $\partial_0 M \times [0,1]$
- takes $M \times 1$ to $N \times 1$
- is transverse to the boundary of $N \times [0,1]$, the inverse image of the boundary being $M \times 0 \cup M \times 1 \cup \partial_0 M \times [0,1]$.

The space of such concordance embeddings is $\mathrm{cemb}(M, N)$. It is not essential here that N be compact. Actually in 3.2.1 and 3.2.2 below we also use concordance embedding spaces $\mathrm{cemb}(M, N \smallsetminus A)$ where A is a closed subset of N, disjoint from M.

The following theorem is a slight reformulation of the main result of [Go1]; see [Go7] for instructions.

3.2.1. Theorem. *If* $m \leq n-3$ *and* $\ell_i \leq n-3$ *for* $i \in S$, *then the contravariant* S*–cube* $\mathrm{cemb}(M, N \smallsetminus L_\bullet)$ *is* $(n-m-2+\Sigma_i \ell'_i)$*–cartesian.*

We state the cases $|S|=0$ and $|S|=1$ explicitly:

3.2.2. Corollary. *If* $m \leq n-3$, *then* $\mathrm{cemb}(M,N)$ *is* $(n-m-3)$*–connected.*

3.2.3. Corollary. *If* $m \leq n-3$ *and* $\ell \leq n-3$, *then the inclusion map* $\mathrm{cemb}(M, N \smallsetminus L) \to \mathrm{cemb}(M,N)$ *is* $(2n-m-\ell-4)$*–connected.*

Corollary 3.2.2 improves on a result due to Hudson [Hu1, Thm. 9.2]. Corollary 3.2.3 is essentially the celebrated Morlet disjunction lemma (Morlet had $2n-m-\ell-4$ only for simply connected N, otherwise $2n-m-\ell-5$). There is no published proof of Morlet's lemma by Morlet, although there were course notes [Mo] at one time. The earliest published proof appears to be the one in [BLR]. For the PL version there is a proof by Millett [Milt1], [Milt2, Thm. 4.2] which uses 'sunny collapsing' (the technique which also Hatcher and Quinn used to prove their disjunction theorem 1.3.1, and which Goodwillie used to prove 3.2.1).

Note that 3.2.3 is not an obvious consequence of a relative version of the Hatcher–Quinn disjunction theorem 1.3.1. There is such a version, but the connectivity estimate we get from it is not good enough. Morlet's lemma is deeper than the Hatcher–Quinn theorem, although it is older. (Conversely, the Hatcher–Quinn theorem is a much better introduction to the subject of disjunction than Morlet's lemma.)

In applications later on, the special case of 3.2.1 where M and the L_i have the same dimension as N is most important. In that case we allow ourselves to mean by $N \smallsetminus L_R$, $N \smallsetminus M$ etc. the *closure* of the complement of L_R, M etc. in N. There is a fibration sequence

$$C(N \smallsetminus M \smallsetminus L_R) \longrightarrow C(N \smallsetminus L_R) \longrightarrow \mathrm{cemb}(M, N \smallsetminus L_R)$$

where C is for spaces of smooth concordances. (A concordance of P is a diffeomorphism $P \times [0,1] \to P \times [0,1]$ restricting to the identity on $\partial P \times [0,1]$ and on $P \times 0$.) From 3.2.2 we also know that $\mathrm{cemb}(M, N \smallsetminus L_R)$ is connected if $m \leq n-3$, in which case we get another homotopy fiber sequence

$$\mathrm{cemb}(M, N \smallsetminus L_R) \longrightarrow BC(N \smallsetminus M \smallsetminus L_R) \longrightarrow BC(N \smallsetminus L_R)\,.$$

Therefore 3.2.1 implies that the diagram $BC(N \smallsetminus M \smallsetminus L_\bullet) \to BC(N \smallsetminus L_\bullet)$ is $(n-m-2+\Sigma_i \ell'_i)$–cartesian. Renaming M as one of the L_i , and enlarging S accordingly, we have:

3.2.4. Corollary. *If* $\ell_i \le n-3$ *for all* $i \in S$, *then* $BC(N \smallsetminus L_\bullet)$ *is* $\Sigma_i \ell'_i$*–cartesian.*

3.3. Multiple disjunction for Poincaré embeddings

Here we assume that N is a Poincaré pair and that M and the L_i for $i \in S$ are Poincaré triads, all of the same formal dimension n. A Poincaré embedding e of L_S in N is fixed. For $R \subset S$ we denote by $N \smallsetminus L_R$ the formal complement of $e|L_R$, viewed as a Poincaré pair.

3.3.1. Theorem. *If* $m \le n-3$ *and* $\ell_i \le n-3$ *for* $i \in S$, *then the diagram* $\mathrm{emb}_{\mathrm{PD}}(M, N \smallsetminus L_\bullet) \longrightarrow \mathrm{map}(M, N \smallsetminus L_\bullet)$ *is* $(n-2m-2+\Sigma_i \ell'_i)$*–cartesian.*

Remarks. The special case $|S| = 1$ is the (codimension zero) Poincaré version of 1.3.1; notice a loss of 1 in the connectivity estimate. In the general form, 3.3.1 is an important ingredient in the proof of 3.5.3 below, a 'multiple' version of 1.3.1, again for smooth embeddings; somewhat miraculously the loss of 1 can be repaired in the deduction.

There is a version of 3.3.1 where M and the L_i are allowed to have arbitrary formal dimensions $\le n$, and where the relative homotopy dimensions m and ℓ_i are replaced by the formal dimensions of M and the L_i. This is an easy consequence of 3.3.1 as it stands.

The full proof of 3.3.1 is still in preparation [GoKl], but a slightly weaker result is proved in [Go6]. Let $H(N \smallsetminus L_R)$ be the space of homotopy automorphisms of $N \smallsetminus L_R$ relative to the boundary. Select a base vertex in $\mathrm{emb}_{\mathrm{PD}}(M, N \smallsetminus L_S)$ if possible. Let

$$\mathcal{X}_R := \mathrm{hofiber}\,[\,H(N \smallsetminus L_R) \xrightarrow{|M} \mathrm{emb}_{\mathrm{PD}}(M, N \smallsetminus L_R)\,]\,,$$
$$\mathcal{Y}_R := \mathrm{hofiber}\,[\,H(N \smallsetminus L_R) \xrightarrow{|M} \mathrm{map}(M, N \smallsetminus L_R)\,]\,.$$

The forgetful arrows $\mathcal{X}_R \to \mathcal{Y}_R$ lead to a diagram $\mathcal{X}_\bullet \to \mathcal{Y}_\bullet$. It is shown in [Go6] that this is $(n-2m-3+\Sigma_i \ell'_i)$–cartesian. The looped version of 3.3.1 follows since the diagram $H(N \smallsetminus L_\bullet) \longrightarrow H(N \smallsetminus L_\bullet)$ given by the identity maps $H(N \smallsetminus L_R) \to H(N \smallsetminus L_R)$ is ∞–cartesian.

3.3.2. Corollary. *If* $m \le n-3$ *and* $\ell_i \le n-3$ *for* $i \in S$, *then the diagram* $\mathrm{emb}_{\mathrm{PD}}(M, N \smallsetminus L_\bullet)$ *is* $(1-m+\Sigma_i \ell'_i)$*–cartesian.*

Sketch proof modulo 3.3.1. The diagram $\mathrm{map}(M, N \smallsetminus L_\bullet)$ is $(1-m+\Sigma_i \ell'_i)$–cartesian. $\square$

Corollary 3.3.2 has a more symmetrical reformulation as a 'higher excision theorem', obtained by renaming M as one of the L_i. (The hypotheses here are a little different: there is no M anymore, since it has been renamed, and no preferred Poincaré embeddings of the L_i in N are specified; but as usual, the $\partial_0 L_i$ are embedded in ∂N.)

3.3.3. Corollary. *If $\ell_i \le n-3$ for $i \in S$, then the diagram* $\mathrm{emb}_{\mathrm{PD}}(L_\bullet, N)$ *is $(3-n+\Sigma_i \ell'_i)$–cartesian.*

Proof modulo 3.3.2. The case $S = \emptyset$ is trivial. Assume $S \neq \emptyset$. Pick $j \in S$. Let $T = S \smallsetminus j$. By [Go3, 1.18] it suffices to show that for every choice of base point e in $\mathrm{emb}_{\mathrm{PD}}(L_T, N)$, the T–cube

$$\mathrm{hofiber}\,[\,\mathrm{emb}_{\mathrm{PD}}(L_{\bullet\cup j}, N) \to \mathrm{emb}_{\mathrm{PD}}(L_\bullet, N)\,]$$

(where $\bullet$ stands for a variable subset of T) is $(3-n+\Sigma_{i\in S}\ell'_i)$–cartesian, in other words $(1-\ell_j+\Sigma_{i\in T}\ell'_i)$–cartesian. But this follows from 3.3.2 (with T in place of S and L_j in place of M), since the homotopy fiber of $\mathrm{emb}_{\mathrm{PD}}(L_{R\cup j}, N) \to \mathrm{emb}_{\mathrm{PD}}(L_R, N)$ over $e|L_R$ is homotopy equivalent to $\mathrm{emb}_{\mathrm{PD}}(L_j\,, N \smallsetminus L_R)$. $\square$

3.4. Higher excision for block embeddings

Here we assume that N and the L_i for $i \in S$ are smooth, all of the same formal dimension n. There is no M.

3.4.1. Theorem. *If $n \ge 5$ and $\ell_i \le n-3$ for $i \in S$, then the diagram* $\mathrm{emb}^\sim(L_\bullet, N)$ *is $(3-n+\Sigma_i \ell'_i)$–cartesian.*

Proof. By 3.3.2 and [Go3, 1.18], it is enough to show that for every choice of base point in $\mathrm{emb}_{\mathrm{PD}}{}^\sim(L_S, N)$, the diagram

$$\mathrm{hofiber}\,[\,\mathrm{emb}^\sim(L_\bullet, N) \to \mathrm{emb}_{\mathrm{PD}}{}^\sim(L_\bullet, N)\,]$$

is $(3-n+\Sigma_i \ell'_i)$–cartesian. But this is ∞–cartesian since, by a mild generalization of 2.2.2, we can identify it with

$$\mathrm{hofiber}\,[\,\mathrm{imm}^\sim(L_\bullet, N) \to \mathrm{imm}_{\mathrm{PD}}{}^\sim(L_\bullet, N)\,]. \quad \square$$

3.5. Higher excision for embeddings

3.5.1. Theorem. *Under the hypotheses of* 3.4.1*, the diagram* $\mathrm{emb}(L_\bullet, N)$ *is* $(3-n+\Sigma_i \ell'_i)$*–cartesian.*

There is an equivalent 'multiple disjunction' version:

3.5.2. Theorem. *If* $n \geq 5$ *and* $n-m \geq 3$*,* $n-\ell_i \geq 3$ *for all* i*, then the cube* $\mathrm{emb}(M, N \smallsetminus L_\bullet)$ *is* $(1-m+\Sigma_i \ell'_i)$*–cartesian.*

Outline of proof of 3.5.1. Choose a base vertex e in $\mathrm{emb}^\sim(L_S, N)$. For $R \subset S$ let $\mathcal{X}_R$ be the homotopy fiber (over $e|L_R$) of the inclusion of $\mathrm{emb}(L_R, N)$ in $\mathrm{emb}^\sim(L_R, N)$. By 3.4.1, it suffices to show that $\mathcal{X}_\bullet$ is $(3-n+\Sigma_i \ell'_i)$–cartesian. There are homotopy fiber sequences

$$\frac{\mathrm{diff}^\sim(N \smallsetminus L_R)}{\mathrm{diff}(N \smallsetminus L_R)} \longrightarrow \frac{\mathrm{diff}^\sim(N)}{\mathrm{diff}(N)} \longrightarrow \mathcal{X}_R$$

where $N \smallsetminus L_R$ is short for the closure of the complement of $e(L_R)$ in N, and all diffeomorphisms in sight restrict to the identity on the appropriate boundary. Therefore (and because $\mathcal{X}_S$ is connected, by 3.2.2) it is enough to show that $\mathcal{Y}(N \smallsetminus L_\bullet)$ is $(2-n+\Sigma_i \ell'_i)$–cartesian, where

$$\mathcal{Y}(P) := \frac{\mathrm{diff}^\sim(P)}{\mathrm{diff}(P)}$$

for a compact smooth P. In fact we will show (twice) that $\mathcal{Y}(N \smallsetminus L_\bullet)$ is $\Sigma_i \ell'_i$–cartesian.

First argument. One of the main results of [WW1], motivated by a spectral sequence due to Hatcher [Hat], says that $\mathcal{Y}(P)$ is, up to homotopy equivalence, the homotopy colimit of a diagram

$$* = F_0\mathcal{Y}(P) \to F_1\mathcal{Y}(P) \to F_2\mathcal{Y}(P) \to \dots$$

where each arrow fits into a homotopy fiber sequence

$$F_j\mathcal{Y}(P) \hookrightarrow F_{j+1}\mathcal{Y}(P) \to B^{j+1}C(P \times [0,1]^j)\,.$$

(Here B^{j+1} denotes $(j+1)$–fold j–connected deloopings.) All of this depends naturally on P, with respect to codimension zero embeddings. Hence it is enough to show that $B^{j+1}C((N \smallsetminus L_\bullet) \times [0,1]^j)$ is $\Sigma \ell'_i$–cartesian, and more than enough to show that it is $(j+\Sigma \ell'_i)$-cartesian. But this follows easily from 3.2.4 (use induction on $|S|$).

Second argument. Again we think of $\mathcal{Y}$ as a functor on compact smooth manifolds and codimension zero embeddings. There is a natural homotopy fiber sequence

$$\mathcal{Y}(P \times [0,1]) \longrightarrow \frac{C^{\sim}(P)}{C(P)} \longrightarrow \mathcal{Y}(P)$$

where $C^{\sim}(P)$, the 'block' version of $C(P)$, is contractible, so that the term in the middle is $BC(P)$. Moreover $\mathcal{Y}(P)$ is connected for any P. These properties are strong enough to imply that the higher excision estimates for the functor BC are also valid for the functor $\mathcal{Y}$. See [Go7] for the details, which are quite elementary. $\square$

3.5.1. Theorem [bis]. *The hypothesis $n \geq 5$ in* 3.5.1 *and* 3.5.2 *is unnecessary.*

Idea of proof. If $n = 3$ then necessarily $m = 0$, so that 3.5.2 for $n = 3$ follows from 3.1.1. Now assume $n = 4$. The looped versions of 3.4.1 and 3.4.4 are then still valid, with the same proof, and for any compatible choice of base points. The looped version of 3.5.2 with $n = 4$ follows, as before, for any choice of base point in $\mathrm{emb}(M, N \smallsetminus L_S)$. Moreover 3.1.1 shows that the diagram in 3.5.2 (but with $n = 4$) is 1–cartesian. This is enough. $\square$

The higher excision theorem 3.5.1 leads to a multiple disjunction theorem for embeddings and maps, in the style of 1.3.1 and 3.3.1. To state it we return to the setup with N, M and L_i for $i \in S$, all of the same dimension n; an embedding $L_S \to N$ is specified.

3.5.3 Theorem. *If $m \leq n-3$ and $\ell_i \leq n-3$ for all $i \in S$, then the diagram* $\mathrm{emb}(M, N \smallsetminus L_\bullet) \to \mathrm{map}(M, N \smallsetminus L_\bullet)$ *is* $(n - 2m - 1 + \Sigma_i \ell_i')$–*cartesian.*

The case $|S| = 1$ of 3.5.3 is the codimension zero case of 1.3.1. Again there exists a version of 3.5.3 where the codimensions of M and the L_i in N are arbitrary. This follows easily from 3.5.3 as it stands.

Idea of proof of 3.5.3. One reduces to the case where M can be obtained from a closed collar on $\partial_0 M$ by attaching a single handle. That case is dealt with by induction on the handle index. (The case where the handle index is zero is trivial.) The induction step uses 3.5.2 and a device called handle splitting. See [Go7, §4,§6] for all details, also [BLR, pf. of 2.3] for handle splitting. — We will indicate another proof (modulo 3.5.1 or 3.5.2) in §4.

4. Calculus methods: Homotopy aspect

In this chapter we approach the 'calculation' of a space of smooth embeddings $\mathrm{emb}(M, N)$ by viewing it as a special value of the cofunctor

$$V \mapsto \mathrm{emb}(V, N)$$

on the poset $\mathcal{O}(M)$ of open subsets V of M. The multiple disjunction and higher excision theorems of chapter 3 imply that if $m \le n-3$, then this cofunctor on $\mathcal{O}(M)$ admits a unique decomposition (Taylor tower) into so–called homogeneous cofunctors, one of each degree $k > 0$. The homogeneous cofunctors are easy to understand and classify. So we end up with something like a functorial calculation of the homotopy type of $\mathrm{emb}(V, N)$, up to extension problems. There is no doubt that the extension problems are serious.

4.1. Taxonomy of cofunctors on $\mathcal{O}(M)$

Let U, V be smooth m–manifolds without boundary. A smooth embedding $e_1 \colon U \to V$ is an *isotopy equivalence* if there exists a smooth embedding $e_2 \colon V \to U$ such that $e_1 e_2$ and $e_2 e_1$ are smoothly isotopic to id_V and id_U , respectively.

4.1.1. Definition. We fix M and write $\mathcal{O} := \mathcal{O}(M)$. A cofunctor F from $\mathcal{O}$ to spaces is *good* if

(i) it takes isotopy equivalences to weak homotopy equivalences (that is, if an inclusion $U \to V$ of open subsets of M is an isotopy equivalence, then the induced map $F(V) \to F(U)$ is a weak homotopy equivalence);

(ii) it takes monotone unions to homotopy inverse limits (that is, if V_i for $i \ge 0$ are open sets in M with $V_i \subset V_{i+1}$, then the canonical map from $F(\bigcup_i V_i)$ to $\mathrm{holim}_i\, F(V_i)$ is a weak homotopy equivalence).

Remark. Call $V \in \mathcal{O}$ *tame* if V is the interior of a compact smooth (codimension zero) submanifold of M. Property (ii) ensures that a good cofunctor F on $\mathcal{O}$ is essentially determined by its behavior on tame open subsets of M. In particular, suppose that F is a cofunctor from $\mathcal{O}(M)$ to spaces having property (i). Then the functor defined by

$$F^\sharp(V) := \operatorname*{holim}_{\text{tame } U \subset V} F(U)$$

for $V \in \mathcal{O}$ is a good cofunctor on $\mathcal{O}$. We call $F^\sharp$ the *taming* of F. Note that $F^\sharp(V) \simeq F(V)$ if V is a tame open subset of M.

4.1.2. Examples. It is not hard to show that the cofunctors given by $V \mapsto \mathrm{emb}(V,N)$, $V \mapsto \mathrm{emb}^{\sim}(V,N)$, $V \mapsto \mathrm{imm}(V,N)$, $V \mapsto \mathrm{imm}^{\sim}(V,N)$ (for fixed smooth N without boundary, and variable V in $\mathcal{O}$) are good. See [We1] for details.

For another example, fix $k \geq 1$, and let $F(V)$ be the space of smooth immersions $g\colon V \to N$ with $|g^{-1}(x)| \leq k$ for all $x \in N$. Then the taming $F^{\sharp}$ of F is good.

4.1.3. Definition. Fix $k \geq 0$. A good cofunctor F on $\mathcal{O}$ is *polynomial of degree* $\leq k$ if, for every $V \in \mathcal{O}$ and pairwise disjoint closed subsets $A_1, \ldots, A_{k+1}$ of V, the $(k+1)$–cube $F(V \smallsetminus A_\bullet)$ is ∞–cartesian. (Here $A_R = \bigcup_{i \in R} A_i$ for a subset R of $\{1, ..., k+1\}$.)

4.1.4. Example. Fix a space X and let $F(V) := \mathrm{map}(V^k, X)$ for $V \in \mathcal{O}$, where V^k means $V \times \cdots \times V$ (k factors). Then F is polynomial of degree $\leq k$. *Idea of proof:* Given V and $A_1, \ldots, A_{k+1}$ as in 4.1.3, one notes using a pigeon hole argument that V^k is the union of the $(V \smallsetminus A_R)^k$ for nonempty $R \subset \{1, ..., k+1\}$. This implies easily that the cubical diagram $(V \smallsetminus A_\bullet)^k$ is ∞–cocartesian. Therefore it turns into an ∞–cartesian diagram when $\mathrm{map}(—, X)$ is applied.

4.1.5. Example. Let $\mathcal{O}k \subset \mathcal{O}$ be the full sub–poset consisting of the V which are diffeomorphic to $\mathbb{R}^m \times S$ with S discrete, $|S| \leq k$. For a good cofunctor F on $\mathcal{O}$, let $T_k F$ be the homotopy right Kan extension (along $\mathcal{O}k \hookrightarrow \mathcal{O}$) of $F|\mathcal{O}k$. Explicitly:

$$T_k F(V) := \operatorname*{holim}_{\substack{W \subset V \\ W \in \mathcal{O}k}} F(W)\,.$$

Then $T_k F$ is again a good cofunctor. The ‘operator’ T_k on good cofunctors comes with an obvious forgetful transformation $\eta_k\colon F(V) \to T_k F(V)$, natural not only in V but also in F. The pair consisting of T_k and η_k has the following properties:

(i) $T_k F$ is polynomial of degree $\leq k$, for any good F.
(i) $\eta_k\colon F(V) \to T_k F(V)$ is a weak homotopy equivalence for all V if F is already polynomial of degree $\leq k$.
(ii) $T_k(\eta_k)\colon T_k F(V) \to T_k(T_k F)(V)$ is (always) a weak homotopy equivalence.

These properties essentially characterize T_k and η_k. One should think of $\eta_k\colon F \to T_k F$ as the best approximation (from the right) of F by a polynomial cofunctor of degree $\le k$. (We also call it the k–th Taylor approximation of F.) In fact, any natural transformation $v\colon F \to G$ where G is polynomial of degree $\le k$ can be enlarged to a commutative square of natural transformations

$$\begin{tikzcd} F \arrow[r, "v"] \arrow[d, "\eta_k"] & G \arrow[d, "\eta_k"] \\ T_kF \arrow[r, "T_kv"] & T_kG \end{tikzcd}$$

where the right–hand column is a natural weak homotopy equivalence by property (i) of T_k and η_k. Thus $v\colon F \to G$ factors through $\eta_k\colon F \to T_kF$, up to formal inversion of a natural weak homotopy equivalence. Property (ii) can be used to show that the factorization is essentially unique (a suitable category of such factorizations has a contractible nerve). See [We1] for all details.

4.1.6. Examples. Suppose that $F(V) = \mathrm{emb}(V, N)$ where N^n is fixed smooth manifold without boundary, and $n > m = \dim(M)$. We will make T_kF explicit for $k = 1$ and $k = 2$. See also 4.3.

Let $F_1(V) := \mathrm{imm}(V, N)$. The natural inclusion $\iota_1\colon F \to F_1$ has the following properties (the first by the immersion classification theorem, the other by inspection):

- the codomain F_1 of ι_1 is polynomial of degree ≤ 1 ;
- ι_1 specializes to a weak homotopy equivalence $F(V) \to F_1(V)$ whenever V is a tubular neighborhood of a single point.

But these two properties of ι_1 essentially characterize $\eta_1\colon F \to T_1F$; so $T_1F(V) \simeq F_1(V) = \mathrm{imm}(V, N)$, by a chain of natural weak homotopy equivalences.

Using the notation from Haefliger's theorem 1.2.1, let $F_2(V)$ be the homotopy pullback (homotopy inverse limit) of the diagram

$$\begin{tikzcd} & \mathrm{map}(V, N) \arrow[d, "f \mapsto f\times f"] \\ \mathrm{ivmap}^{\mathbb{Z}/2}(V \times V, N \times N) \arrow[r, "\subset"] & \mathrm{map}^{\mathbb{Z}/2}(V \times V, N \times N)\,. \end{tikzcd}$$

Then there is a forgetful natural transformation $\iota_2\colon F \to F_2$. One checks easily that

- the codomain F_2 of ι_2 is polynomial of degree ≤ 2 ;
- ι_2 specializes to a weak homotopy equivalence $F(V) \to F_2(V)$ whenever V is a tubular neighborhood of a subset S of M of cardinality ≤ 2.

(The homotopy inverse limit of a diagram of good cofunctors on $\mathcal{O}$ which are polynomial of degree $\le k$ is again polynomial of degree $\le k$. Therefore, to show that F_2 is polynomial of degree ≤ 2, it suffices to show that the cofunctors

$$V \mapsto \mathrm{map}(V, N)$$
$$V \mapsto \mathrm{map}^{\mathbb{Z}/2}(V \times V, N \times N)$$
$$V \mapsto \mathrm{ivmap}^{\mathbb{Z}/2}(V \times V, N \times N)$$

are polynomial of degree $\le 1, 2, 2$ respectively, and this can be done much as in 4.1.4.) These properties of ι_2 essentially characterize $\eta_2\colon F \to T_2F$, and it follows that $T_2F(V) \simeq F_2(V)$ by a chain of natural weak homotopy equivalences.

4.1.7. Definition. A good cofunctor F on $\mathcal{O}$ is *homogeneous of degree* k if it is polynomial of degree $\le k$ and if $T_{k-1}F(V)$ is weakly homotopy equivalent to a point, for all $V \in \mathcal{O}$.

4.1.8. Example. Let $\binom{M}{k}$ be the space of unordered configurations of k distinct points in M. Let

$$p\colon E \to \binom{M}{k}$$

be a fibration. Suppose that this is equipped with the structure of a germ σ of partial sections, defined 'near' the fat diagonal (complement of $\binom{M}{k}$ in the space of unordered k–tuples of points in M). For $V \in \mathcal{O}$ let $F(V)$ be the space of partial sections of p which are defined on $\binom{V}{k}$ and agree with σ near the fat diagonal. Then F is a good cofunctor which is homogeneous of degree k. There is a classification theorem for homogeneous cofunctors on $\mathcal{O}$ which says that they can all be obtained in this way (up to a natural weak homotopy equivalence), from a pair (p, σ) as above, unique up to fiber homotopy equivalence respecting section germs. We call p the classifying fibration of the homogeneous cofunctor.

If F is any good cofunctor on $\mathcal{O}$, with a preferred base point in $F(M)$, then L_kF defined by

$$L_kF(V) := \mathrm{hofiber}\,[\,T_kF(V) \xrightarrow{\text{forget}} T_{k-1}F(V)\,]$$

is a homogeneous cofunctor of degree k. Its classifying fibration p on $\binom{M}{k}$ must have a preferred global section σ, corresponding to the base point of $L_kF(M)$. The fibration p and the global section σ can be described roughly as follows. For $S \subset M$ with $|S| = k$, and each $x \in S$, choose a small open ball V_x about x. For $R \subset S$ let $V_R = \cup_{x \in R} V_x$. Then the fiber of p over S is the total homotopy fiber of the contravariant S–cube $R \mapsto F(V_R)$. Note that this is a pointed space.

4.1.9. Example. Let $F(V) = \mathrm{emb}(V, N)$. Fix a base point in $F(M)$, alias embedding $M \to N$. We describe the classifying fibration(s) p_k for L_kF, any $k > 0$, simplifying the general description in 4.1.8 as much as possible. First, p_1 is the forgetful map and fibration

$$E_1 \longrightarrow M$$

where $E_1 = \{(x, z, f) \mid x \in M, z \in N,\ f\colon T_xM \to T_zN \text{ linear injective}\}$. Second, p_k for $k > 1$ is the fibration

$$E_k \longrightarrow \binom{M}{k}$$

whose fiber over $S \in \binom{M}{k}$ is the total homotopy fiber of the cubical diagram of pointed spaces given by $R \mapsto \mathrm{emb}(R, N)$ for $R \subset S$. (These spaces are pointed because $R \subset S \subset M \subset N$.) To see that these are correct descriptions, make a forgetful map, between spaces over $\binom{M}{k}$, from the standard description of p_k (classifying fibration for L_kF) as given in 4.1.8 to the new description under scrutiny; then verify that it is a fiberwise homotopy equivalence.

4.1.10. Definition. Let F be a good cofunctor F on $\mathcal{O}$. We say that F is *ρ–analytic with excess c* (where $\rho, c \in \mathbb{Z}$) if it has the following property. For $V \in \mathcal{O}$ and $k > 0$ and pairwise disjoint closed subsets $A_1, \dots, A_{k+1}$ of V, where each A_i is a smooth submanifold of V of codimension $q_i < \rho$, diffeomorphic to euclidean space, the cube $F(V \smallsetminus A_\bullet)$ is $(c + \Sigma_i(\rho - q_i))$–cartesian.

Remark. To motivate 4.1.10 just a little, we note that the definition of a polynomial cofunctor, 4.1.3, can be reformulated as follows. A good cofunctor F on $\mathcal{O}$ is polynomial of degree $\leq k$ if it has the following property. For $V \in \mathcal{O}$ and pairwise disjoint closed subsets $A_1, \dots, A_{k+1}$ of V, where each A_i is a smooth submanifold of V, diffeomorphic to a euclidean space, the cube $F(V \smallsetminus A_\bullet)$ is ∞–cartesian.

Indeed, if F has the property, then $F(V \smallsetminus B_\bullet)$ will be ∞–cartesian whenever $V \in \mathcal{O}$ is tame, $B_1, \dots, B_{k+1}$ are pairwise disjoint closed subsets of V, and the closure $\bar{B}_i$ of B_i in M is a compact codimension zero smooth manifold triad embedded in $\bar{V}$, with $\partial_0 \bar{B}_i = \bar{B}_i \cap \partial \bar{V}$ and $\partial_1 \bar{B}_i$ transverse to $\partial \bar{V}$. The proof is by an easy (multiple) induction over the number of handles required to build each $\bar{B}_i$ from a collar on $\partial_0 \bar{B}_i$. An application of the limit axiom for good cofunctors then shows that $F(V \smallsetminus C_\bullet)$ will be ∞–cartesian whenever $V \in \mathcal{O}$ is tame, and $C_1, \dots, C_{k+1}$ are pairwise disjoint closed subsets of V.

4.1.11. Digression/Definition. Given a finite set S and an S–cube $\mathcal{X}$ of spaces and $z \in \mathbb{R}$, let us say that $\mathcal{X}$ is z–cartesian if the canonical map

$$\mathcal{X}(\emptyset) \longrightarrow \operatorname*{holim}_{\emptyset \neq R \subset S} \mathcal{X}(R)$$

has connectivity $\geq z$. With this convention, 4.1.10 remains meaningful for arbitrary $\rho, c \in \mathbb{R}$. This will become important in §5.

4.1.12. Definitions. The theory has a variant where M is a manifold with boundary, and F is a cofunctor on $\mathcal{O}(M)$, the poset of all open subsets of M containing ∂M. The kind of functor we have in mind is $V \mapsto \operatorname{emb}(V, N)$ where N is fixed, with boundary, and an embedding $e \colon \partial M \to \partial N$ has been specified. In the definition of $\operatorname{emb}(V, N)$ we allow only embeddings $V \to N$ which agree with e on ∂V, and are transverse to ∂M.

A good cofunctor F from $\mathcal{O}(M)$ to spaces is *polynomial of degree* $\leq k$ if $F(V \smallsetminus A_\bullet)$ is ∞–cartesian for any $V \in \mathcal{O}(M)$ and pairwise disjoint subsets $A_0, \dots, A_k$ of V, closed in V and disjoint from ∂M. The k–th Taylor approximation $T_k F$ of an arbitrary good cofunctor F on $\mathcal{O}(M)$ is defined by

$$T_k F(V) := \operatorname*{holim}_{\substack{W \in \mathcal{O}k \\ W \subset V}} F(W)$$

where $\mathcal{O}k = \mathcal{O}k(M)$ consists of the $W \in \mathcal{O}(M)$ which are tubular neighborhoods of $\partial M \cup S$ for some subset S of $M \smallsetminus \partial M$, with $|S| \leq k$. A homogeneous functor F of degree k on $\mathcal{O}(M)$ has a *classifying fibration*

$$p \colon E \longrightarrow \binom{M}{k}$$

equipped with a germ σ of sections, defined near fat diagonal *and* on the boundary. Then $F(V)$ is, up to a chain of natural homotopy equivalences, the space of (partial) sections of p defined over $\binom{V}{k}$ and agreeing with σ near the fat diagonal and on the boundary. The classifying fibration p_k for the k–th homogeneous layer, $k \geq 2$, of the cofunctor $V \mapsto \mathrm{emb}(V, N)$, as above, has fiber $p_k^{-1}(S)$ equal to the total homotopy fiber of the cube

$$R \mapsto \mathrm{emb}(R, N) \qquad (R \subset S)\,.$$

4.2. The convergence theorem

The *Taylor tower* of a good cofunctor F on $\mathcal{O}$ is the diagram of good cofunctors and (forgetful) transformations

$$\cdots \xrightarrow{r_{k+1}} T_k F \xrightarrow{r_k} T_{k-1}F \xrightarrow{r_{k-1}} T_{k-2}F \xrightarrow{r_{k-2}} \cdots\,.$$

It should be regarded as a diagram of cofunctors *under* F, since for each k we have $\eta_k\colon F \to T_k F$ and the relations $r_k \eta_k = \eta_{k-1}$ hold.

4.2.1. Theorem. *Suppose that F is ρ–analytic with excess c, and $V \in \mathcal{O}$ has a proper Morse function with critical points of index $\leq q$ only, where $q < \rho$. Then the connectivity of*

$$\eta_{k-1}\colon F(V) \longrightarrow T_{k-1}F(V)$$

is $\geq c + k(\rho - q)$, for $k > 1$. Therefore $F(V) \xrightarrow{\simeq} \mathrm{holim}_k\, T_k F(V)$. In words, the Taylor tower of F, evaluated at V, converges to $F(V)$.

See [GoWe, 2.3] for the proof, which is quite easy. Although originally intended for the situation where $\rho, c \in \mathbb{Z}$, it goes through with arbitrary $\rho, c \in \mathbb{R}$. Compare 4.1.11.

4.2.2. Corollary. *If F is ρ–analytic, and $\rho > m = \dim(M)$, then $F(V) \simeq \mathrm{holim}_k\, T_k F(V)$ for all $V \in \mathcal{O}$.*

4.2.3. Theorem–Example. *Let $F(V) = \mathrm{emb}(V, N)$ for $V \in \mathcal{O}$, where N^n is fixed (smooth, without boundary). Then F is $(n-2)$–analytic with excess $3 - n$.*

Idea of proof. Fix a finite set S. It suffices to check that $F(V \smallsetminus A_\bullet)$ is $(3 - n + \Sigma_i(n - q_i - 2))$–cartesian if

- $V \in \mathcal{O}$ is tame;
- $A_i = D_i \cap V$ for $i \in S$, where $D_i \subset \bar{V}$ is a smoothly embedded disk of codimension $q_i < n - 2$, transverse to the boundary of $\bar{V}$, with $\partial D_i = D_i \cap \partial\bar{V}$, and the D_i are pairwise disjoint.

Next, fix some smooth embedding $e\colon V \smallsetminus A_S \to N$. It is enough to show that the cube

$$\mathrm{hofiber}_e\,[\,F(V \smallsetminus A_\bullet) \longrightarrow F(V \smallsetminus A_S)\,]$$

is $(3 - n + \Sigma_{i\in S}(n - q_i - 2))$–cartesian. We can assume that e extends to a smooth embedding $\bar{e}\colon \bar{V} \to N$, and further, to a codimension zero embedding $f\colon W \to N$ where $W \to \bar{V}$ is the disk bundle of the normal bundle of $\bar{e}$. Let N' be the closure in N of the complement of $f(W)$. Then

$$\mathrm{hofiber}_e\,[\,F(V \smallsetminus A_R) \longrightarrow F(V \smallsetminus A_S)\,]$$

is naturally homotopy equivalent to $\mathrm{emb}(D_R, N')$ where $D_R = \cup_{i\in R} D_i$ for $R \subset S$. (Note that preferred embeddings $\partial D_i \to \partial N'$ are given.) Hence it is enough to show that the cube $\mathrm{emb}(D_\bullet, N')$ is $(3-n+\Sigma_{i\in S}(n-q_i-2))$–cartesian. But this follows from 3.5.1 (actually, the 'arbitrary codimension' version of 3.5.1). □

4.2.4. Corollary/Summary. *Let $F(V) = \mathrm{emb}(V, N)$, and assume that the codimension $n - m$ is ≥ 3. Suppose for simplicity $M \subset N$, so that each $F(V)$ is a based space. Then*

$$\eta_{k-1}\colon F(V) \longrightarrow T_{k-1}F(V)$$

is $(3 - n + k(n - m - 2))$–connected; therefore $F(V) \xrightarrow{\simeq} \mathrm{holim}_k\, T_kF(V)$. We have $T_1F(V) \simeq \mathrm{imm}(V, N)$. For $k > 1$, the homotopy fiber $L_kF(V)$ of $T_kF(V) \to T_{k-1}F(V)$ is homotopy equivalent to the space of sections, vanishing near the fat diagonal, of

$$p_k\colon E_k \longrightarrow \binom{M}{k}$$

where $p_k^{-1}(S)$ for $S \in \binom{M}{k}$ is the total homotopy fiber of the S–cube defined by $R \mapsto \mathrm{emb}(R, N)$ for $R \subset S$.

Remark. There is a considerable shortcut to corollary 4.2.4 in the cases where $2m < n - 2$. In those cases we can avoid most of chapter 3, using only the easy higher excision theorem, in the symmetric form 3.1.2, to show that F is $(n-m-2)$–analytic. Since $m < n-m-2$, this implies according to 4.2.2 that

$$F(V) \xrightarrow{\simeq} \operatorname*{holim}_k T_kF(V)$$

for all $V \in \mathcal{O}$. The analysis of the layers $L_kF(V)$ goes through as before. We can now use 4.1.9 and again 3.1.2 to show that the fibers of the classifying fibration for L_kF are $(k+1)(n-2)$–connected; hence $L_kF(V)$ is $((k+1)(n-2)-mk)$–connected and $T_kF(V) \to T_{k-1}F(V)$ is $((k+1)(n-2)-mk+1)$–connected, i.e., $(3-n+k(n-m-2))$–connected. It follows that η_k from $F(V) \simeq \mathrm{holim}_k\, T_kF(V)$ to $T_{k-1}F(V)$ is $(3-n+k(n-m-2))$–connected. $\square$

4.2.5. Example. This example is meant to illustrate the 'with boundary' variant of 4.2.4. Suppose that $M=[0,1]$ and that N has a boundary, and $M \subset N$ as a submanifold, ∂M being the transverse intersection of M with ∂N. Let $F(V) := \mathrm{emb}(V,N)$ as in 4.1.12. Then

$$\binom{M}{k} \cong \Delta^k$$

and so the k–th homogeneous layer $L_kF(M)$ becomes the k–th loop space of any of the fibers of the classifying fibration for L_kF. If in addition N is homotopy equivalent to a suspension, $N \simeq \Sigma Y$, then this can be analyzed with the Hilton-Milnor theorem, and one finds

$$L_kF(M) \simeq \prod_w{}' \Omega^k \Sigma^{1+\alpha(w)(n-2)} Y^{(\beta(w))}$$

for $k>1$, where the weak product $\prod'$ is over all basic words w in the letters $z_1,\dots,z_k$ involving all letters except possibly z_1. See [GoWe, §5] for more details and explanations.

4.2.6. Remark. Two different calculus approaches to block embedding spaces $\mathrm{emb}^{\sim}(M,N)$ come to mind. One of these is to view $\mathrm{emb}^{\sim}(M,N)$ as a special value of a good cofunctor F on $\mathcal{O}(M)$, and to approximate it by the $(T_rF)(M)$ for $r \ge 0$. The other is to think of $\mathrm{emb}^{\sim}(M,N)$ as the geometric realization of a simplicial space

$$k \mapsto \mathrm{emb}^{\sim}_{\dots}(M \times \Delta^k,\, N \times \Delta^k)$$

where the dots indicate certain boundary conditions; then, to view each $\mathrm{emb}^{\sim}_{\dots}(M \times \Delta^k,\, N \times \Delta^k)$ as a special value of a cofunctor F_k defined on the open subsets of $M \times \Delta^k$; then, to approximate $\mathrm{emb}^{\sim}(M,N)$ by the geometric realizations of

$$k \mapsto (T_rF_k)(M \times \Delta^k)$$

for $r \ge 0$, where T_rF_k is a suitable Taylor approximation to F_k which we have not defined and will not define here.

Taking $M = *$ shows that these approaches give quite different results. The second appears to be superior. It is still very much under construction, and so we will not waste more words on it, except by saying that it sheds light on the Levine problem (section 2.3). In fact it has been used, in the quadratic alias metastable range, in an unpublished paper by Larmore and Williams [LW] on the Levine problem. Their main result, which they prove without surgery, is the generalization of 2.3.2 to the situation where the domain M is compact, smooth, but not necessarily closed.

4.3. Scanning revisited

Let $F(V) = \mathrm{emb}(V, N)$ as in 4.1.6, 4.1.9, 4.2.3, 4.2.4. Our goal here is to give a description of the Taylor approximation $F \to T_k F$, for $k \geq 2$, which generalizes the Haefligeresque description of $F \to T_2 F$ in 1.2.1 and 4.1.6.

Notation. Think of the standard $(k-1)$–simplex Δ^{k-1} as an incomplete simplicial set whose i–simplices are the monotone injections z from $\{0, ..., i\}$ to $\{1, ..., k\}$. With such an i–simplex z we can associate the set $\{1, ..., z(i)\}$, filtered by subsets $\{1, ..., z(j)\}$ for $0 \leq j \leq i$. Let $G(z)$ be the group of permutations of $\{1, ..., z(i)\}$ which respect the filtration, and let $G_0(z)$ be the full permutation group of $\{1, ..., z(0)\}$, so that $G_0(z)$ is a factor in an obvious product decomposition of $G(z)$. Write $[z := z(0)$ and $z] := z(i)$ where $i = |z|$.

4.3.1. Definition. For $k \geq 2$ and a simplex z of Δ^{k-1}, let $J_{M,N,k}(z) = J_M(z)$ be the space of smooth maps

$$M^{z]} \longrightarrow N^{[z}$$

which are strongly isovariant with respect to $G_0(z)$, and equivariant with respect to $G(z)$. (The actions of $G_0(z)$ on $M^{z]}$ and $N^{[z}$ are by permutation of the coordinates labeled 1 through $[z$. The action of $G(z)$ on $M^{z]}$ is by permutation of the coordinates labeled 1 through $z]$. The action of $G(z)$ on $N^{[z}$ is obtained from the action of $G_0(z)$ on $N^{[z}$ just defined by means of the projection $G(z) \to G_0(z)$.)

Then $J_M(z)$ is a functor of the variable z. (If y is a face of z, then we have homomorphisms $G(z) \to G(y)$ and $G_0(z) \hookrightarrow G_0(y)$, and we also have projections $N^{[y} \to N^{[z}$, $M^{z]} \to M^{y]}$ which are both $G(z)$–equivariant and strongly $G_0(z)$–isovariant.)

4.3.2. Definition. We let $\Theta_k(M,N) = \mathrm{holim}_z\, J_M(z)$ with J_M as in definition 4.3.1. Explicitly, $\Theta_k(M,N)$ is the space of natural transformations from the functor $z \mapsto \Delta^{|z|}$ to the functor $z \mapsto J_M(z)$.

Motivation. Let $\mathcal{D}_k(M)$ be the topological poset of functions $g\colon M \to \mathbb{N}$ with finite support, and degree $|g| := \sum_x g(x)$ satisfying $1 \le |g| \le k$. Here $\mathbb{N} = \{0,1,2,...\}$; for $f, g \in \mathcal{D}_k(M)$ we decree $g \le f$ if $g(x) \le f(x)$ for all $x \in M$, and we topologize $\mathcal{D}_k(M)$ by identifying it with the coproduct of the M^i/Σ_i for $1 \le i \le k$.

For $g \in \mathcal{D}_k(M)$ let $p(g)$ be the support, a subset of M of cardinality between 1 and k. The idea is that $\Theta_k(M,N)$ is a modified version of the topological homotopy limit of the functor $g \mapsto \mathrm{emb}(p(g),N)$. The expression *topological homotopy limit* indicates that we pay attention to the topological structure of $\mathcal{D}_k(M)$. The modification happens where we ask for strongly isovariant smooth maps rather than just isovariant continuous maps.

4.3.3. Example. Let $k=2$. Let's denote the simplices of Δ^1 by $I, 0, 1$ in this case. Let $f = \{f_I, f_0, f_1\}$ be any point in $\Theta_2(M,N)$. Then f_1 is a strongly isovariant Σ_2–map from M^2 to N^2, and f_0 is just a smooth map $M \to N$. Finally f_I is a path (parametrized by $[0,1]$) of smooth maps $M^2 \to N$. Its values at time 1 and 0 respectively are the compositions

$$M^2 \xrightarrow{f_1} N^2 \longrightarrow N$$

$$M^2 \longrightarrow M \xrightarrow{f_0} N.$$

It follows that $\Theta_2(M,N)$ is (homeomorphic to) the Haefliger approximation to $\mathrm{emb}(M,N)$ of 1.2.1 and 4.1.6.

4.3.4. Theorem. *$\Theta_k(M,N) \simeq T_k\,\mathrm{emb}(M,N)$, for $k \ge 2$.*

Idea of proof. Let $F(V) = \mathrm{emb}(V,N)$ for $V \in \mathcal{O}$. We will show that $T_kF(V)$ is naturally weakly homotopy equivalent to $\Theta_k(V,N)$. There is a natural inclusion $F(V) \to \Theta_k(V,N)$. It suffices to show that

(i) $\Theta_k(V,N)$ is polynomial of degree $\le k$ as a functor of V ;
(ii) the natural inclusion $F(V) \to \Theta_k(V,N)$ is a homotopy equivalence whenever V is in $\mathcal{O}k$.

To establish (i) it is enough to show that each of the functors $V \mapsto J_V(z)$ is polynomial of degree $\leq k$. This is easy. For (ii), suppose that V is a tubular neighborhood of $S \subset M$, where $|S| \leq k$. One checks that

$$\begin{array}{ccc} \mathrm{emb}(V,N) & \longrightarrow & \Theta_k(V,N) \\ \Big\downarrow \text{res.} & & \Big\downarrow \text{res.} \\ \mathrm{emb}(S,N) & \longrightarrow & \Theta_k(S,N) \end{array}$$

is ∞–cartesian. With the motivation above, it is not hard to show that $\mathrm{emb}(S,N) \to \Theta_k(S,N)$ is a homotopy equivalence. See [GoKW] for the details. $\square$

5. Calculus methods: Homology aspect

5.1. One–dimensional domains

One of us (Goodwillie) observed long ago that when $M = I = [0,1]$, compare 4.1.12, the calculus of good cofunctors F on $\mathcal{O}(M)$ amounts to a theory of cosimplicial spaces and their corealizations (corealization = Tot). It can therefore give *homological* information about $F(M) = F(I)$ (which tends to play the role of the corealization) by means of the generalized Eilenberg–Moore spectral sequence [Bou], [Re], [EM], the standard tool for calculating the homology of such corealizations. These ideas are explained here. Following Bott [Bo], we make contact with the theory of knot invariants of finite type initiated by Vassiliev [Va1], [Va2], [Va3], [BiL], [BaN], [BaNSt], [Ko], [Bi] and extensions of it used by Kontsevich [Ko] in his calculation of $H^*(\mathrm{emb}(\mathbb{S}^1, \mathbb{R}^n);\mathbb{Q})$ for $n > 3$.

Let $\mathcal{O} = \mathcal{O}(I)$ and $\mathcal{O}k = \mathcal{O}k(I)$, with the conventions of 4.1.12. We want to establish a correspondence between good cofunctors from $\mathcal{O}$ to spaces, and *augmented* cosimplicial spaces, that is, covariant functors from the category of all finite totally ordered sets (including the empty set) to spaces. Let $\mathcal{O}' \subset \mathcal{O}$ consist of all elements which have only finitely many connected components, so that

$$\mathcal{O}' = \{I\} \cup \bigcup_{k \geq 0} \mathcal{O}k\,.$$

A good cofunctor on $\mathcal{O}$ is determined up to natural weak homotopy equivalence by its restriction to $\mathcal{O}'$. The restriction is still an isotopy invariant cofunctor. Hence it is enough to establish a correspondence between isotopy invariant cofunctors from $\mathcal{O}'$ to spaces, and augmented cosimplicial spaces. (We write augmented cosimplicial spaces in the form $S \mapsto \mathfrak{F}_S$, or in the form $\mathfrak{F}_\emptyset \to \mathfrak{F}_\bullet$. Here the bullet stands for a nonempty finite totally ordered set, so that $\mathfrak{F}_\bullet$ is the underlying un–augmented cosimplicial space.)

5.1.1. Constructions. Let κ be the cofunctor from $\mathcal{O}'$ to totally ordered finite sets given by $V \mapsto \pi_0(I \smallsetminus V)$. Pre–composition with κ gets us from augmented cosimplicial spaces to isotopy invariant space–valued cofunctors on $\mathcal{O}'$. Conversely, an isotopy invariant cofunctor F from $\mathcal{O}'$ to spaces determines an augmented cosimplicial space by homotopy right Kan extension along κ,

$$\mathfrak{F}_S := \operatorname*{holim}_{V \text{ with } S \to \kappa(V)} F(V)$$

for a finite totally ordered S. These two construction are inverses of one another, up to natural weak homotopy equivalence.

5.1.2. Definitions. Let $\mathfrak{F}_\bullet$ be any cosimplicial space. For $0 \le k \le \infty$ let $\mathrm{Tot}^k(\mathfrak{F}_\bullet)$ be the space of natural transformations from $S \mapsto \Delta(S)$ to $S \mapsto \mathfrak{F}_S$, for totally ordered finite S with $1 \le |S| \le k$. Here $\Delta(S)$ denotes the simplex spanned by S. When $k = \infty$, we simply write $\mathrm{Tot}(\mathfrak{F}_\bullet)$, and speak of the corealization. There is a tower of forgetful maps (Serre fibrations)

$$\mathrm{Tot}(\mathfrak{F}_\bullet) \cdots \to \mathrm{Tot}^k(\mathfrak{F}_\bullet) \to \mathrm{Tot}^{k-1}(\mathfrak{F}_\bullet) \to \cdots \to \mathrm{Tot}^0(\mathfrak{F}_\bullet)\,.$$

Let $\mathfrak{C}_\bullet$ be a cosimplicial chain complex. For $0 \le k \le \infty$ let $\mathrm{Tot}^k(\mathfrak{C}_\bullet)$ be the chain complex of natural maps of graded abelian groups from $S \mapsto C_*(\Delta_S)$ to $S \mapsto \mathfrak{C}_S$, for totally ordered finite S with $1 \le |S| < \infty$, where C_* is the singular chain complex functor. (The i–chains in $\mathrm{Tot}^k(\mathfrak{C}_\bullet)$ are the natural maps raising degrees by i, for $i \in \mathbb{Z}$.) When $k = \infty$, we write $\mathrm{Tot}(\mathfrak{C}_\bullet)$. There is a tower of chain complexes and forgetful chain maps

$$\mathrm{Tot}(\mathfrak{C}_\bullet) \cdots \to \mathrm{Tot}^k(\mathfrak{C}_\bullet) \to \mathrm{Tot}^{k-1}(\mathfrak{C}_\bullet) \to \cdots \to \mathrm{Tot}^0(\mathfrak{C}_\bullet)\,.$$

Each of these chain maps is a ‘fibration’ (degreewise split onto). With such a tower of fibrations of chain complexes, one can associate in the usual way an exact couple and/or a spectral sequence converging, under mild conditions on $\mathfrak{C}_\bullet$, to the homology of $\mathrm{Tot}(\mathfrak{C}_\bullet)$.

In particular, suppose that $\mathfrak{C}_\bullet = C_*(\mathfrak{F}_\bullet)$ is the cosimplicial chain complex obtained from a cosimplicial space by applying C_*. Then under suitable conditions on $\mathfrak{F}_\bullet$,

- the spectral sequence converges to $H_* \operatorname{Tot}(C_*(\mathfrak{F}_\bullet))$, and
- the canonical map $H_* \operatorname{Tot}(\mathfrak{F}_\bullet) \to H_* \operatorname{Tot}(C_*(\mathfrak{F}_\bullet))$ is an isomorphism.

In that case we can say simply that the spectral sequence converges to $H_* \operatorname{Tot}(\mathfrak{F}_\bullet)$. It is called a 'generalized Eilenberg–Moore spectral sequence' because, according to Rector [Re], the original Eilenberg–Moore spectral sequence [EM] for the calculation of the homology of a homotopy pullback of spaces is a special case.

5.1.3. Remark. Let $\mathcal{A}$ be an abelian category. The Dold–Kan correspondence [Cu] is an equivalence of categories, often denoted N for 'normalization', from simplicial $\mathcal{A}$–objects to chain complexes in $\mathcal{A}$ graded over the integers ≥ 0. In particular, the Dold–Kan correspondence associates to a cosimplicial chain complex $\mathfrak{C}_\bullet$ a cochain complex $N\mathfrak{C}_\bullet$ of chain complexes

$$N\mathfrak{C}_0 \xrightarrow{d_0} N\mathfrak{C}_1 \xrightarrow{d_0} N\mathfrak{C}_2 \xrightarrow{d_0} \cdots .$$

Here each $N\mathfrak{C}_i$ is a chain complex in its own right, the quotient of $\mathfrak{C}_i$ by the chain subcomplex generated by the images of the face operators $d_j\colon \mathfrak{C}_{i-1} \to \mathfrak{C}_i$ for $0 < j \leq i$. It is also (as a chain complex) a direct summand of $\mathfrak{C}_i$. Now $\operatorname{Tot}^k(\mathfrak{C}_\bullet)$ is isomorphic to the 'total chain complex' [CaE] of the truncated double complex

$$N\mathfrak{C}_0 \to N\mathfrak{C}_1 \to \cdots \to N\mathfrak{C}_k .$$

Although this does not help much in explaining the generalized Eilenberg–Moore spectral sequence above, where $\mathfrak{C}_\bullet = C_*(\mathfrak{F}_\bullet)$, it does lead to the insight that the E^1 and E^2–terms are

$$\begin{aligned} E^1_{-p,q} &\cong N^p(H_q\mathfrak{F}_\bullet), \\ E^2_{-p,q} &\cong H^p(N(H_q\mathfrak{F}_\bullet)) . \end{aligned}$$

Here $H_q\mathfrak{F}_\bullet$ for fixed q is a cosimplicial abelian group, and $N(H_q\mathfrak{F}_\bullet)$ is the associated cochain complex, with p–th cochain group $N^p(H_q\mathfrak{F}_\bullet)$. The spectral sequence lives in the second quadrant. With these grading conventions, the differentials on E^r have bidegree $(-r, r-1)$, and $E^\infty_{-p,q}$ is (in the convergent case) a subquotient of $H_{q-p}(\operatorname{Tot}\mathfrak{F}_\bullet)$.

Now suppose that $\mathfrak{F}_\emptyset \to \mathfrak{F}_\bullet$ is the augmented cosimplicial space associated with a good cofunctor F from $\mathcal{O} = \mathcal{O}(I)$ to spaces. Then it follows easily from the definitions that

$$\begin{aligned} \mathfrak{F}_S &\simeq F(I \smallsetminus S) \qquad\qquad \text{for finite } S \subset I \smallsetminus \partial I, \\ \operatorname{Tot}^k \mathfrak{F}_\bullet &\simeq T_k F(I), \\ \operatorname{Tot} \mathfrak{F}_\bullet &\simeq \operatorname*{holim}_k T_k F(I)\,. \end{aligned}$$

Under these identifications the comparison map $F(I) \to \operatorname{holim}_k T_k F(I)$ corresponds to the augmentation–induced map $\mathfrak{F}_\emptyset \to \operatorname{Tot} \mathfrak{F}_\bullet$. In particular, if F is ρ–analytic with $\rho > 1$, then by the convergence theorem

$$\mathfrak{F}_\emptyset \quad \xrightarrow{\ \simeq\ } \quad \operatorname{Tot} \mathfrak{F}_\bullet\,.$$

Therefore, assuming Bousfield's convergence criteria [Bou] are satisfied, the spectral sequence constructed above converges to $H_* F(I)$; more precisely, we can write

$$\{\, E^2_{-p,q} = H^p(N(H_q F(I \smallsetminus \bullet)))\,\} \qquad \Rightarrow \qquad \{\, H_{q-p} F(I)\,\}$$

where $\bullet$ runs through a selection of nonempty finite subsets of $I \smallsetminus \partial I$, one for each (finite, nonzero) cardinality.

5.1.4. Example. For $V \in \mathcal{O}$ let $F(V)$ be the homotopy fiber of the inclusion $\operatorname{emb}(V, \mathbb{R}^{n-1} \times I) \hookrightarrow \operatorname{imm}(V, \mathbb{R}^{n-1} \times I)$, where $n \geq 3$. Boundary conditions as in 4.1.12 are understood. Note that $\operatorname{imm}(V, \mathbb{R}^{n-1} \times I)$ is homotopy equivalent to the space of pointed maps from $V/\partial V$ to $\mathbb{S}^{n-1}$ by immersion theory. — The generalized Eilenberg–Moore spectral sequence has

$$E^2_{-p,q} \;\cong\; H^p(N(H_q(\operatorname{emb}(\{1,2,...,\bullet\}, \mathbb{R}^n))))$$

where $\bullet$ runs through the integers ≥ 0. The homology of the 'configuration space' $\operatorname{emb}(\{1,2,\ldots,k\}, \mathbb{R}^n)$ is torsion free, therefore dual to the cohomology of $\operatorname{emb}(\{1,\ldots,k\}, \mathbb{R}^n)$. The cohomology ring $H^*(\operatorname{emb}(\{1,\ldots,k\}, \mathbb{R}^n))$ is the quotient of an exterior algebra on generators α_{st} in degree $n-1$, one such for any two distinct elements $s, t \in \{1,\ldots,k\}$, by relations

$$\begin{gathered} \alpha_{st} = (-1)^n \alpha_{ts}\,, \\ \alpha_{rs}\alpha_{st} + \alpha_{st}\alpha_{tr} + \alpha_{tr}\alpha_{rs} = 0\,. \end{gathered}$$

Here α_{st} is the image of the canonical generator under the map in cohomology induced by

$$\operatorname{emb}(\{1,2,\ldots,k\}, \mathbb{R}^n) \longrightarrow \mathbb{R}^n \smallsetminus 0 \quad ; \quad g \mapsto g(t) - g(s).$$

These assertions can be proved by induction on k, using the fact that the Leray–Serre spectral sequence associated with the forgetful fibration $\mathrm{emb}(\{1,\dots,k\},\mathbb{R}^n) \to \mathrm{emb}(\{1,\dots,k-1\},\mathbb{R}^n)$ collapses at E_2. Our description of $H_*(\mathrm{emb}(\{1,\dots,k\},\mathbb{R}^n))$ is so natural that it is in fact a description of the cosimplicial graded abelian group $H_*(\mathrm{emb}(\{1,\dots,\bullet\},\mathbb{R}^n))$, thereby delivering $E^2_{-p,q} \cong H^p(N(H_q(\mathrm{emb}(\{1,2,...,\bullet\},\mathbb{R}^n))))$, the E^2–term of the Eilenberg–Moore spectral sequence. We omit the details, but mention the following points.

(i) When $n > 3$, Bousfield's convergence condition [Bou, Thm.3.4] is satisfied; we will verify this somewhat indirectly in 5.2 below. Therefore the spectral sequence converges to the homology of

$$F(I) = \mathrm{hofiber}\,[\,\mathrm{emb}(I,\mathbb{R}^{n-1}\times I) \to \mathrm{imm}(I,\mathbb{R}^{n-1}\times I)\,]\,.$$

It seems to be very closely related to a spectral sequence developed by Kontsevich in [Ko], for the calculation of the rational cohomology of $\mathrm{emb}(\mathbb{S}^1,\mathbb{R}^n)$ where $n > 3$. However, Kontsevich can also show that his spectral sequence collapses.

(ii) When $n = 3$, the set $\pi_0 F(I)$ can be identified with the set of framed knots in $\mathbb{R}^3$ which are regularly homotopic as framed immersions to the standard one. So we are doing knot theory. — The pieces of the E^1–term of the spectral sequence in total degree < 0 vanish, by inspection. Hence, for the pieces in total degree 0, there are surjections

$$E^1_{-p,p} \to E^2_{-p,p} \to E^3_{-p,p} \to E^4_{-p,p} \to \cdots\ .$$

For odd p we have $E^1_{-p,p} = 0$. For even p, the term $E^1_{-p,p}$ is isomorphic to the free abelian group generated by the set of partitions of $\{1,\dots,p\}$ into $p/2$ subsets of cardinality 2. The relations introduced in passing to $E^2_{-p,p}$ can be calculated from the above information. They are

$$u\cdot\gamma \sim 0, \qquad v\cdot\gamma \sim 0\,,$$

where γ is a generator corresponding to a partition containing two parts of the form $\{r,s\}$ and $\{s+1,t+1\}$ with $r<s<t$, and u,v are certain elements in the group ring of the symmetric group Σ_p (which acts by pushforward). Namely,

$$u = 1-(s,s{+}1)+(t{+}1,t,\dots,s)-(t,t{+}1)(t{+}1,t,\dots,s)\,,$$
$$v = 1-(s,s{+}1)+(r,r{+}1,\dots,s{+}1)-(r,r{+}1)(r,r{+}1,\dots,s{+}1).$$

The reader familiar with the theory of knot invariants of finite type [Va1],[Va2], [Va3], [BiL], [Ko], [BaN], [Bi] will now recognize $E^2_{-p,p}$ as the degree $p/2$ part of $\mathcal{A}$, the graded algebra of chord diagrams modulo the so–called 4T relation; see particularly [BaN].

As Bott points out in [Bo], this suggests that passage from $H_0F(I)$ to $H_0\operatorname{Tot}\mathfrak{F}_\bullet$ and subsequent analysis of $H_0\operatorname{Tot}\mathfrak{F}_\bullet$ by means of the spectral sequence is an alternative approach to the theory of (framed) knot invariants of finite type. However, as Bott also points out, it is far from obvious that the surjections

$$E^2_{-p,p} \longrightarrow E^\infty_{-p,p}$$

are bijections (and consequently we do not have a straightforward construction of framed knot invariants in $\mathcal{A}$ using this approach). If they are, we expect that any proof will use substantial parts of the existing theory of knot invariants of finite type, such as the Kontsevich integrals [Ko], [BaN].

5.2. Higher dimensional domains

One conclusion to be drawn from 5.1 is that the notion of an isotopy invariant cofunctor F from $\bigcup_{k\ge 0}\mathcal{O}k(M)$ to spaces is a legitimate generalization of the notion of cosimplicial space (special case $M = I = [0,1]$). In particular, the construction $F \mapsto \operatorname{holim} F$ is the correct generalization of Tot, and $F \mapsto \operatorname{holim}(F|\mathcal{O}k(M))$ is the correct generalization of Tot^k. The Eilenberg–Moore–Rector–Bousfield question of whether Tot commutes with 'linearization' functors from spaces to spaces

$$\lambda_{\boldsymbol{J}}\colon X \mapsto \Omega^\infty(X_+ \wedge \boldsymbol{J})$$

(where $\boldsymbol{J}$ denotes a fixed CW–spectrum) turns into the question of whether $\lambda_{\boldsymbol{J}}(\operatorname{holim} F) \simeq \operatorname{holim} \lambda_{\boldsymbol{J}}F$. But we already have a conditional answer to the generalized question. Namely, if F is defined on all of $\mathcal{O}(M)$, and sufficiently analytic, and if $\lambda_{\boldsymbol{J}}F$ is also sufficiently analytic on $\mathcal{O}(M)$, then we will have

$$\begin{array}{ccccc} F(M) & \xrightarrow{\simeq} & \operatorname*{holim}_k T_kF(M) & \xrightarrow{\simeq} & \operatorname*{holim}_{V\in\bigcup\mathcal{O}k(M)} F(V)\,, \\ \lambda_{\boldsymbol{J}}F(M) & \xrightarrow{\simeq} & \operatorname*{holim}_k T_k(\lambda_{\boldsymbol{J}}F)(M) & \xrightarrow{\simeq} & \operatorname*{holim}_{V\in\bigcup\mathcal{O}k(M)} \lambda_{\boldsymbol{J}}F(V)\,. \end{array}$$

We then also have a (twice generalized) Eilenberg–Moore type spectral sequence converging to the homotopy of $\lambda_{\boldsymbol{J}}F(M)$, which is essentially the

$\boldsymbol{J}$–homology of $F(M)$. It is the homotopy spectral associated with the tower

$$\cdots \to T_{k+1}(\lambda_{\boldsymbol{J}}F)(M) \to T_k(\lambda_{\boldsymbol{J}}F)(M) \to T_{k-1}(\lambda_{\boldsymbol{J}}F)(M) \to \cdots$$

where $T_k(\lambda_{\boldsymbol{J}}F)(V)$ is nothing but $\operatorname{holim}(\lambda_{\boldsymbol{J}}F|\mathcal{O}k(M))$. From 4.1.8, we have quite a good understanding of its E^1–term. Of course, we do *not* claim that $T_k(\lambda_{\boldsymbol{J}}F)$ agrees in any sensible sense with $\lambda_{\boldsymbol{J}}(T_kF)$, except as it were for $k=\infty$ by Eilenberg–Moore type magic.

In the following lemma M^m is arbitrary (smooth, possibly with boundary). If there is a nonempty boundary, define $\mathcal{O}(M)$ as in 4.1.12. For the first time we use the generalization 4.1.11 of definition 4.1.10 of an analytic cofunctor.

5.2.1. Lemma. *Let F be a good cofunctor on $\mathcal{O}(M)$ and let $\boldsymbol{J}$ be a (-1)–connected CW–spectrum. Suppose that $T_{r-1}F \simeq *$ for some $r>0$, and F is ρ–analytic with excess $c<0$, where $\rho + c/r > m$. Then the taming of $\lambda_{\boldsymbol{J}}F$ is $(\rho + c/r)$–analytic with excess 0.*

See [We2] for the proof.

5.2.2. Example. Let M be compact, oriented, and $M^m \subset N^n$ as a smooth submanifold, $\partial M = M \cap \partial N$ (transverse intersection). Let

$$F(V) = \operatorname{hofiber}[\operatorname{emb}(V,N) \to \operatorname{imm}(V,N)]$$

with conventions as in 4.1.12. Then F is $(n-2)$–analytic with excess $3-n$, by 4.2.3 and §3 of [GoWe]. Applying 5.2.1 with $r=2$ and $\boldsymbol{J} = \boldsymbol{H}\mathbb{Z}$, and writing λ for $\lambda_{\boldsymbol{H}\mathbb{Z}}$, we find that the taming of λF is $(n/2-1/2)$–analytic with excess 0, provided $n/2 - 1/2 > m$. In that case the connectivity of the Taylor approximations

$$\lambda F(M) \longrightarrow T_k(\lambda F)(M)$$

tends to infinity as $k \to \infty$. Then the spectral sequence determined by the exact couple $(E^1, D^1, \dots)$ with

$$D^1_{-p,q} := \pi_{q-p}(T_{p-1}(\lambda F)(M)),$$
$$E^1_{-p,q} := \pi_{q-p}[T_p(\lambda F)(M) \longrightarrow T_{p-1}(\lambda F)(M)] = \pi_{q-p}L_p(\lambda F)(M)$$

converges to $\{\pi_{q-p}(\lambda F(M))\} = \{H_{q-p}(F(M))\}$. Its E^1–term simplifies by 4.1.8 and Poincaré duality to

$$E^1_{-p,q} \quad \cong \quad \begin{cases} 0 & (p < 2) \\ H_{pm-1-q}(Y(M,N,p)\,;\,\mathbb{Z}^{\pm}) & (p \geq 2) \end{cases}$$

where $Y(M,N,p)$ is the space over $\binom{M}{p}$ whose fiber over $S \in \binom{M}{p}$ is

$$\operatorname*{hocolim}_{\emptyset \neq R \subset S} \text{hofiber}\,[\,\text{emb}(R,N) \to N^R\,].$$

When m is odd, untwisted integer coefficients $\mathbb{Z}^+$ are understood; when m is even, use $\mathbb{Z}^-$, integer coefficients twisted by means of the composition

$$\pi_1 Y(M,N,p) \to \Sigma_p \to \mathbb{Z}/2 = \text{aut}(\mathbb{Z})\,.$$

Acknowledgment

M. Weiss enjoyed the hospitality and stimulating atmosphere of the Max–Planck Institute for Mathematics, Bonn, in Fall of 1998 and Spring of 1999, when parts of this survey came into being. We also wish to thank Greg Arone for help and advice related to chapter 5.

References

[BaN] D. Bar–Natan, *On the Vassiliev knot invariants*, Topology **34** (1995), 423–472.

[BaNSt] D. Bar–Natan and A.Stoimenow, *The fundamental theorem of Vassiliev invariants*, 1995 Aarhus Conf. Geometry and Physics, Lecture Notes in Pure and Appl. Math., 184, Dekker, NewYork, 1997, pp. 101–134.

[BaW] M.G. Barratt and J.H.C. Whitehead, *The first non–vanishing group of an $(n+1)$–ad*, Proc. Lond. Math. Soc. **6** (1956), 417–439.

[BH] I.Berstein and P.J.Hilton, *On suspensions and comultiplications*, Topology **2** (1963), 73–82.

[Bi] J. Birman, *New points of view in knot theory*, Bull. Amer. Math. Soc. **28** (1993), 253–287.

[BiL] J. Birman and X.-S. Lin, *Knot polynomials and Vassiliev's invariants*, Invent. Math. **111** (1993), 225–270.

[BlM1] A.L. Blakers and W.S. Massey, *The homotopy groups of a triad. I*, Ann. of Math. **53** (1951), 161–205.

[BlM2] A.L. Blakers and W.S. Massey, *The homotopy groups of a triad. II*, Ann. of Math. **55** (1952), 192–201.

[BS] J. M. Boardman and B. Steer, *On Hopf invariants*, Comment Math. Helv. **42** (1967), 180–221.

[Bo] R. Bott, *On invariants of manifolds*, 1992 Princeton Conf. on modern methods in complex analysis, Ann.of Math. Studies vol. 137, Princeton University Press, Princeton, New Jersey, 1995, pp. 29-39.

[Bou] A.K.Bousfield, *On the homology spectral sequence of a cosimplicial space*, Amer. J.of Math. **109** (1987), 361–394.

[BK] A.K.Bousfield and D.Kan, *Homotopy limits, completions and localizations*, Lect. Notes in Math., vol. 304, Springer, 1973.

[Br1] W. Browder, *Embedding* 1*–connected manifolds*, Bull. Amer. Math. Soc. **72** (1966), 225–231 and 736.

[Br2] W. Browder, *Embedding smooth manifolds*, Proc. ICM Moscow 1966, Mir, 1968, pp. 712–719.

[Br3] W. Browder, *Poincaré spaces, their normal fibrations, and surgery*, Invent. Math. 17 (1972), 191–202.

[BLR] D. Burghelea, R. Lashof, M. Rothenberg, *Groups of Automorphisms of Manifolds*, Springer Lect. Notes in Math. vol. 473, Springer–Verlag, 1975.

[Ca] A.J.Casson, *Fibrations over spheres*, Topology **6** (1967), 489–499.

[CaE] H.Cartan and S.Eilenberg, *Homological algebra*, Princeton University Press, Princeton, New Jersey, 1956.

[Cu] E.B.Curtis, *Simplicial homotopy theory*, Adv. in Math. **6** (1971), 107–209.

[Da] J.-P.Dax, *Etude homotopique des espaces de plongements*, Ann. Scient. de l'École Norm. Sup. **5** (1972), 303–377.

[Dr] E.Dror–Farjoun, *Homotopy and homology of diagrams of spaces*, 1985 Seattle conf. on Algebraic Topology, Springer Lect. Notes in Math., vol. 1286, Springer, 1987.

[DwK1] W.Dwyer and D.Kan, *A classification theorem for diagrams of simplicial sets*, Topology **23** (1984), 139–155.

[DwK2] W.Dwyer and D.Kan, *Equivalences between homotopy theories of diagrams*, 1983 Princeton conference on Algebraic Topology and Algebraic K–Theory, Ann. of Math. Studies, vol. 113, Princeton Univ. Press, 1987.

[EM] S.Eilenberg and J.C.Moore, *Homology and fibrations. I. Coalge-*

bras, cotensor product and its derived functors, Comment. Math. Helv. **40** (1966), 199–236.

[ES] G.Ellis and R.Steiner, *Higher dimensional crossed modules and the homotopy groups of* $(n+1)$*-ads*, J.Pure and Appl. Algebra **46** (1987), 117–136.

[Ga1] T. Ganea, *A generalization of the homology and homotopy suspension*, Comment. Math. Helv. **39** (1965), 295–322.

[Ga2] T. Ganea, *Induced fibrations and cofibrations*, Trans. Amer. Math. Soc. **127** (1967), 442–459.

[Ga3] T. Ganea, *On the homotopy suspension*, Comment. Math. Helv. **43** (1968), 225–234.

[Go1] T. Goodwillie, *A multiple disjunction lemma for smooth concordance embeddings*, Memoirs of the Amer. Math. Soc. vol. 86, no. 431, 1990.

[Go2] T. Goodwillie, *Calculus I: The first derivative of pseudoisotopy theory*, K–theory **4** (1990), 1–27.

[Go3] T. Goodwillie, *Calculus II: Analytic functors*, K–theory **5** (1992), 295–332.

[Go4] T. Goodwillie, *Calculus III: The Taylor series of a homotopy functor*, in preparation.

[Go5] T. Goodwillie, *The differential calculus of homotopy functors*, Proceedings of the International Congress of Mathematicians, Vol. I,II, Kyoto 1990, Math. Soc. Japan, Tokyo, pp. 621–630.

[Go6] T. Goodwillie, *Excision estimates for spaces of homotopy equivalences*, preprint 1995.

[Go7] T. Goodwillie, *Excision estimates for spaces of diffeomorphisms*, preprint 1998.

[GoKl] T. Goodwillie and J. Klein, *Excision estimates for spaces of Poincaré embeddings* (tentative title), in preparation.

[GoKW] T. Goodwillie, J. Klein and M. Weiss, *A Haefliger type description of the embedding calculus tower*, Preprint 1999, available from `http://maths.abdn.ac.uk/~mweiss`.

[GoWe] T. Goodwillie and M. Weiss, *Embeddings from the point of view of immersion theory, Part II*, Geometry and Topology **3** (1999), 103–118.

[Hae1] A. Haefliger, *Plongements differentiables de varietes dans varietes*, Commentarii Math. Helv. **36** (1962), 47–82.

[Hae2] A.Haefliger, *Plongements différentiables dans le domaine stable*, Commentarii Math. Helv. **37** (1962/63), 155–167.

[Hae3] A. Haefliger, *Lectures on the theorem of Gromov*, Proc. of Liverpool 1969/70 Singularities Symp. II, Lecture Notes in Math. vol. 209, Springer, 1971, pp. 128–141.

[Hae4] A. Haefliger, *Differentiable embeddings of $\mathbb{S}^n$ in $\mathbb{S}^{n+q}$ for $q > 2$*, Ann. of Math. **83** (1966), 402–436.

[HaeH] A.Haefliger and M.Hirsch, *Immersions in the stable range*, Ann. of Math. **75** (1962), 231–241.

[Hat] A. E. Hatcher, *Concordance spaces, higher simple homotopy theory, and applications*, Proc. Symp. Pure math., vol. 32 part I, 1978, pp. 3–21.

[HaQ] A.E. Hatcher and F.Quinn, *Bordism invariants of intersections of submanifolds*, Trans. Amer. Math. Soc. **200** (1974), 327–344.

[Hi1] M.Hirsch, *Immersions of manifolds*, Trans. Amer. Math. Soc **93** (1959), 242–276.

[Hi2] M.Hirsch, *Embeddings and compressions of smooth manifolds and polyhedra*, Topology **4** (1966), 361–369.

[Ho] J.P.E.Hodgson, *Obstructions to concordance for thickenings*, Invent. Math. **5** (1968), 292–316.

[Hu1] J.F.P.Hudson, *Piecewise Linear Topology*, Benjamin, New York, Amsterdam, 1969.

[Hu2] J.F.P.Hudson, *Embeddings of bounded manifolds*, Proc. of the Camb. Phil. Soc. **72** (1972), 11–25.

[Jm] I.M.James, *The topology of Stiefel manifolds*, London Mathematical Society Lecture note series vol. 24, Cambridge University Press, 1976.

[Kln1] J.R.Klein, *Poincaré embeddings and fiberwise homotopy theory*, Topology **38** (1999), 597–620.

[Kln2] J.R.Klein, *Poincaré embeddings and fiberwise homotopy theory, II*, in preparation.

[Kln3] J.R.Klein, *Embedding, compression, and fiberwise homotopy theory*, Preprint.

[Ko] M.Kontsevich, *Feynman Diagrams and Low–dimensional Topology*, Proceedings of First European Congress of Mathematics (1992), vol. II, Birkhäuser, pp. 97–121.

[LW] L. Larmore and B. Williams, *Single obstructions to embedding and Boardman–Vogt little cubes*, Preprint 1978?, available from Bruce Williams at Notre Dame University.

[Lau1] F. Laudenbach, *Disjonction de sous–variétés et application au croisement des anses*, Ann. Scient École Norm. Sup. **3** (1970), 385–408.

[Lau2] F. Laudenbach, *Disjonction homotopique et disjonction isotopique: la première obstruction*, C.R. Acad. Sc. **273** (1971), 1037–1039.

[Lau3] F. Laudenbach, *Disjonction homotopique et disjonction isoto-*

pique: la première obstruction, Ann. Scient. École Norm. Sup. **5** (1972), 397–422.

[Lev] J.Levine, *On differentiable embeddings of simply connected manifolds*, Bull. Amer. Math. Soc. **72** (1963), 806–809.

[Lt] N.Levitt, *Normal fibrations for complexes*, Ill. J. of Math. **14** (1970), 385–408.

[LLZ] Li Banghe, Liu Rong, Zhang Ping, *A simplified approach to embedding problems in normal bordism framework*, Systems Science and Math. Sciences **5** (1992), 180–192.

[MaL] S.MacLane, *Categories for the working mathematician*, Graduate texts in Math., vol. 5, Springer, 1971.

[Milt1] K. Millett, *Piecewise linear concordances and isotopies*, Memoirs Amer. Math. Soc., vol. 153, Amer. Math. Soc., 1974.

[Milt2] K. Millett, *Piecewise linear embeddings of manifolds*, Illinois J. of Math **19** (1975), 354-369.

[Mi1] J.Milnor, *Lectures on the h–cobordism theorem*, Notes by L. Siebenmann and J.Sondow, Mathematical Notes 1, Princeton Univ. Press, 1965.

[Mo] C.Morlet, *Plongements et automorphismes des varietes*, Notes, Cours Peccot, College de France, Paris (1969).

[Ra] A.Ranicki, *The algebraic theory of surgery, II. Applications to topology*, Proc. Lond. Math. Soc. **40** (1980), 87–192.

[Re] D.Rector, *Steenrod operations in the Eilenberg–Moore spectral sequence*, Comment. Math. Helv. **45** (1970), 540–552.

[Ric] W.Richter, *A homotopy theoretic proof of Williams' metastable Poincaré embedding theorem*, Duke Math. Jour. **88** (1997), 435–447.

[RiW] R.Rigdon and B.Williams, *Embeddings and Immersions of Manifolds*, Proc. of 1977 Evanston Conf. on Geometric Applications of Homotopy Theory, I, Springer Lect. Notes in Math. vol. 657, 1978, pp. 423-454.

[Sa] H.A.Salomonsen, *On the existence and classification of differential embeddings in the metastable range*, Aarhus mimeographed notes, 1973.

[Sha] A. Shapiro, *Obstructions to the imbedding of a complex in a euclidean space*, Ann. of Math. **66** (1957), 256–269.

[Sm1] S. Smale, *The classification of immersions of spheres in Euclidean spaces*, Ann. of Math. **69** (1959), 327–344.

[Sta] J. Stallings, *Lectures on polyhedral topology*, Tata Institute Research Lectures on Math., vol. 43, Tata Institute of Fundamental Research, Bombay, 1967.

[Ti] A. Tinéo, *Un théorème de relèvement dans les espaces d'applications différentiables; réalisation d'intersection*, Thesis, Univ. de Paris, Orsay, 1971.

[van] Van Kampen, Abh. Math. Sem. Hamburger Universität **9** (1932), 72–78, 152–153.

[Va1] V.A.Vassiliev, *Cohomology of knot spaces*, Theory of singularities and its applications, Adv. Soviet Math., 1, Amer.Math.Soc., Providence, RI, 1990.

[Va2] V.A.Vassiliev, *Complements of discriminants of smooth maps: topology and applications*, Translations of Math. Monographs, vol. 98, Amer. Math. Soc., Providence, RI, 1992.

[Va3] V.A.Vassiliev, *Invariants of knots and complements of discriminants*, Developments in Mathematics: the Moscow school, Chapman and Hall, London, 1993, pp. 194–250.

[Wa1] C.T.C. Wall, *Geometric connectivity. I*, J. Lon. Math. Soc **3** (1971), 597–604.

[Wa2] C.T.C. Wall, *Surgery on Compact Manifolds*, Acad. Press, London–New York, 1970; 2nd edition, Amer. Math. Soc., 1999.

[We1] M.Weiss, *Embeddings from the point of view of immersion theory, Part I*, Geometry and Topology **3** (1999), 67–101.

[We2] M. Weiss, *Homology of spaces of smooth embeddings*, Preprint, University of Aberdeen, 1999, currently available from `http://maths.abdn.ac.uk/~mweiss`.

[We3] M.Weiss, *Calculus of Embeddings*, Bull. Amer. Math. Soc. **33** (1996), 177–187.

[WW1] M.Weiss and B.Williams, *Automorphisms of Manifolds and Algebraic K–theory: I*, K-Theory **1** (1988), 575–626.

[Wh1] H.Whitney, *Differentiable manifolds*, Ann. of Math. **37** (1936), 645–680.

[Wh1] H.Whitney, *The self–intersections of a smooth n–manifold in $2n$–space*, Ann. of Math. **45** (1944), 220–246.

[Wi1] B. Williams, *Applications of unstable normal invariants — Part I*, Compositio Math. **38** (1979), 55–66.

[Wi2] B. Williams, *Hopf invariants, localization, and embeddings of Poincaré complexes*, Pacific J. Math. **84** (1979).

[Ze] E.C. Zeeman, *Unknotting combinatorial balls*, Ann. of Math. **78** (1963), 501–526.

Dept. of Mathematics
Brown University
Providence RI 02912, USA

E-mail address: tomg@math.brown.edu

Dept. of Mathematics
Wayne State University
Detroit MI 48202, USA
E-mail address: klein@math.wayne.edu

Dept. of Mathematics
University of Aberdeen
Aberdeen AB24 3UE, U.K.
E-mail address: m.weiss@maths.abdn.ac.uk