

Lecture 1.

Seiberg–Witten theory: preliminaries

DIRAC OPERATORS (the local theory)

Factorizations of the Laplace operator

Let $C^\infty(U; \mathbb{C}^N)$ be the linear space of the smooth $\mathbb{C}^N$ -valued functions on an open set $U \subset \mathbb{R}^n$, and

$$\Delta : C^\infty(U; \mathbb{C}^N) \longrightarrow C^\infty(U; \mathbb{C}^N), \quad \Delta = - \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2},$$

the Laplace operator. Any first order differential operator

$$D : C^\infty(U; \mathbb{C}^N) \longrightarrow C^\infty(U; \mathbb{C}^N)$$

such that $D^2 = \Delta$ is called a (local) **Dirac operator**.

Let $D = \sum_{k=1}^n A_k \frac{\partial}{\partial x_k}$ where A_k are complex $(N \times N)$ -matrices. Then

$$\begin{aligned} D^2 &= \sum_{k, \ell} A_k \frac{\partial}{\partial x_k} (A_\ell \frac{\partial}{\partial x_\ell}) \\ &= \sum_{k, \ell} A_k A_\ell \frac{\partial^2}{\partial x_k \partial x_\ell} \\ &= \sum_k A_k^2 \frac{\partial^2}{\partial x_k^2} + \sum_{k < \ell} (A_k A_\ell + A_\ell A_k) \frac{\partial^2}{\partial x_k \partial x_\ell}. \end{aligned}$$

and one can easily see that the condition $D^2 = \Delta$ implies that

$$\begin{cases} A_k^2 = -1 & \text{for all } k, \text{ and} \\ A_k A_\ell + A_\ell A_k = 0 & \text{for all } k \neq \ell \end{cases} \quad (1.1)$$

As soon as we have n matrices $A_1, \dots, A_n$ satisfying the conditions (1.1), we have a Dirac operator.

Examples

n=1. Take $N = 1$ and $A_1 = i$, then $D = i \frac{\partial}{\partial x}$ is a Dirac operator.

n=2. Let $N = 2$ and A_1 and A_2 are the matrices

$$A_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

n=3 ('Pauli spin matrices') One can take $N = 2$ again, then A_1 and A_2 can be chosen as above, and

$$A_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

To generalize these simple examples to arbitrary dimensions we need to take a closer look at the nature of the matrices $A_1, \dots, A_n$.

Representation theory of Clifford algebras

Let $e_1, \dots, e_n$ be an orthonormal basis in $\mathbb{R}^n$. **Clifford algebra** $Cl(\mathbb{R}^n)$ is the $\mathbb{R}$ -algebra generated by $e_1, \dots, e_n$ with the relations $e_k^2 = -1$ for any $k \leq n$, and $e_k e_\ell + e_\ell e_k = 0$ for any $k \neq \ell$. Any $\mathbb{C}$ -linear representation

$$\rho : Cl(\mathbb{R}^n) \longrightarrow \text{End}(S)$$

of this algebra yields a set of matrices $\rho(e_k)$, $k = 1, \dots, n$, satisfying the relations (1.1) (the complex linear space S plays the role of $\mathbb{C}^N$ above), and vice versa, any set $A_1, \dots, A_n$ of $(N \times N)$ -matrices satisfying (1.1) uniquely determines a representation ρ by putting $S = \mathbb{C}^N$ and $\rho(e_k) = A_k$. So, we want to understand the representation theory of $Cl(\mathbb{R}^n)$. It turns out that our conclusions will depend on whether n is odd or even (as a matter of fact, we are mostly interested in $n = 3$ and $n = 4$).

We shall analyse the representation theory of $Cl(\mathbb{R}^n)$ by looking at the representation theory of a finite multiplicative subgroup. Let E be the group of order 2^{n+1} consisting of all the elements

$$\pm e_1^{\varepsilon_1} e_2^{\varepsilon_2} \cdots e_n^{\varepsilon_n}$$

where each of $\varepsilon_1, \dots, \varepsilon_n$ is either 0 or 1 (check that this is indeed a group). In particular, E contains $(-1) \in \mathbb{R}$; denote this by ν when it is considered as an element of E .

Proposition 1.2. *There is a one-to-one correspondence between representations of $Cl(\mathbb{R}^n)$ and those of E on which ν acts as $-I$.*

Proof. Obvious. The condition on ν comes from the fact that for any linear representation ρ of $Cl(\mathbb{R}^n)$, we must have $\rho(\nu) = \rho(e_k^2) = (\rho(e_k))^2 = -I$. $\square$

Thus, one can use the representation theory of the finite group E to study that of $Cl(\mathbb{R}^n)$. We need the following three theorems about representations of finite groups.

Theorem 1.3. *All irreducible representations of a finite abelian group are 1-dimensional, and their number equals the order of the group.*

Theorem 1.4. *The number of irreducible representations of a finite group is equal to the number of the conjugacy classes in the group.*

Theorem 1.5. *Let G be a finite group, then the sum of squares of dimensions of its irreducible representations equals the order of the group.*

Now, the centre of the group E consists of 1, ν , $\eta = e_1 \cdots e_n$, and $\nu\eta$ in the case of odd n , and only of 1 and ν when n is even. It can be seen as follows. Let $g = e_1^{\varepsilon_1} \cdots e_n^{\varepsilon_n}$ be central. Suppose that there exist r and s such that $\varepsilon_r = 1$, $\varepsilon_s = 0$, then one can check by hand that $e_r e_s g = \nu g e_r e_s$, and we get a contradiction. Therefore, all ε_k 's should be simultaneously 0 or 1, and the only possible central elements are 1, ν , $\eta = e_1 \cdots e_n$, and $\nu\eta$. But if $n = 2m$ is even we have

$$e_1 \eta = \nu \eta e_1,$$

so the claim follows.

Since ν is a central involution in the finite group E , we must have $\rho(\nu) = \pm 1$ for any irreducible representation ρ of E . Those irreducible representations ρ for which $\rho(\nu) = 1$ are representations of the abelian group $E/\{1, \nu\}$ of order 2^n , so according to Theorem 1.3 there are 2^n of them. How many more representations does E have?

We begin with the case of even n . Let us count the conjugacy classes in E . The conjugacy class of $g \in E$ must be either $\{g\}$ (if g is central) or $\{g, \nu g\}$ (otherwise). This is an easy consequence of the fact that $E/\{1, \nu\}$ is abelian. The number of conjugacy classes is therefore

$$\frac{2^{n+1} - 2}{2} + 2 = 2^n + 1.$$

Due to Theorem 1.4, so is the number of irreducible representations of E . It follows that E has, up to conjugation, just one more irreducible representation ρ , and $\rho(\nu)$ must be -1 . The corresponding representation of the Clifford algebra is called the **spin representation**. Its dimension can be calculated by using Theorem 1.5. Namely,

$$2^n + (\dim \rho)^2 = 2^{n+1}.$$

Therefore, $\dim \rho = 2^{n/2}$ (remember that n is even).

Now assume that $n \equiv 3 \pmod{4}$. A simple computation shows that $\eta^2 = 1$ in this case and that η is independent of the choice of oriented orthogonal basis. The left multiplication by $-\eta$ gives a canonical decomposition of $Cl(\mathbb{R}^n)$ into the ± 1 eigenspaces

$$Cl(\mathbb{R}^n) = Cl(\mathbb{R}^n)^+ \oplus Cl(\mathbb{R}^n)^-. \quad (1.6)$$

Since $-\eta$ is contained in the centre of $Cl(\mathbb{R}^n)$, this means that $Cl(\mathbb{R}^n)^\pm$ are subalgebras which annihilate each other and (1.6) is an orthogonal sum of algebras.

Lemma 1.7. *If $n \equiv 3 \pmod{4}$ then the algebras $Cl(\mathbb{R}^n)^\pm$ are both isomorphic to $Cl(\mathbb{R}^{n-1})$.*

Proof. The required isomorphisms of algebras are obtained as the compositions

$$Cl(\mathbb{R}^{n-1}) \xrightarrow{[e_n]} Cl(\mathbb{R}^n) \xrightarrow{\pi^\pm} Cl(\mathbb{R}^n)^\pm,$$

where the embedding $[e_n]$ is generated by putting $e_k \mapsto e_n e_k$ for each k , with $e_1, \dots, e_{n-1}$ being an orthonormal basis of $\mathbb{R}^{n-1}$, and $\pi^\pm$ are the obvious projections. The maps $\pi^\pm$ are expressed as left multiplications by $(1 \mp \eta)/2$ and the lemma follows as the image of $[e_n]$ in $Cl(\mathbb{R}^n)$ meets both $Cl(\mathbb{R}^n)^\pm$ trivially. $\square$

Notice that the embedding map $[e_n]$ is the graph of an algebra isomorphism $Cl(\mathbb{R}^n)^+ \rightarrow Cl(\mathbb{R}^n)^-$.

Remark. The decomposition (1.6) and Lemma 1.7 generalize to any odd n if one considers the complex form $Cl(\mathbb{R}^n) \otimes \mathbb{C}$ of Clifford algebra and replaces the element $-\eta$ with $i^{[\frac{n+1}{2}]} e_1 \cdots e_n$.

The subalgebra of $Cl(\mathbb{R}^n)$ obtained as the image of $[e_n]$ makes sense for any n . This is the **even part of Clifford algebra**, denoted $Cl_0(\mathbb{R}^n)$, defined as a linear span of

the basis elements $e_{j_1} \cdots e_{j_k}$, where k is even. Respectively, by taking k odd one defines the **odd part** $Cl_1(\mathbb{R}^n)$ of $Cl(\mathbb{R}^n)$ and there is a $\mathbb{Z}_2$ -grading

$$Cl(\mathbb{R}^n) = Cl_0(\mathbb{R}^n) \oplus Cl_1(\mathbb{R}^n),$$

and $Cl_0(\mathbb{R}^n) \cong Cl(\mathbb{R}^{n-1})$.

In the case of odd n , the number of conjugacy classes in E is $(2^{n+1} - 4)/2 + 4 = 2^n + 2$, so $Cl(\mathbb{R}^n)$ then has exactly two non-isomorphic irreducible, finite dimensional, complex representations up to isomorphism. If n is congruent to 3 modulo 4, they are obtained by projecting onto one of the two factors in (1.6) and taking the spin representation of that factor. Since $Cl(\mathbb{R}^{n-1})$ in this case is embedded diagonally in the direct sum, it is clear that the restrictions of the non-isomorphic irreducible complex representations to $Cl_0(\mathbb{R}^n) \cong Cl(\mathbb{R}^{n-1})$ become isomorphic. In particular, any of these two representations of $Cl(\mathbb{R}^n)$ has dimension $2^{\frac{n-1}{2}}$ (which can also be verified by elementary number theory methods, considering the constraint provided by Theorem 1.5). Note that whereas the two representations coincide on the even part $Cl_0(\mathbb{R}^n)$ they have opposite signs on the odd part $Cl_1(\mathbb{R}^n)$, in particular, on the generators $e_1, \dots, e_n$.

The spin representation $\rho : Cl(\mathbb{R}^{2m}) \rightarrow \text{End}(S)$ can be explicitly constructed as follows. Denote by V the real $2m$ -dimensional Euclidean vector space, one can always assume that there exists a linear operator $J : V \rightarrow V$ such that $J^2 = -I$ compatible with the Euclidean structure, $(Jx, Jy) = (x, y)$. Then $V \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$ where

$$T^{1,0} = \{t \in V \otimes \mathbb{C} \mid Jt = it\}$$

$$T^{0,1} = \{t \in V \otimes \mathbb{C} \mid Jt = -it\}$$

are the $\pm i$ eigenspaces of J acting on $V \otimes \mathbb{C}$ as $J \otimes I$. We also define a Hermitian metric on $V \otimes \mathbb{C}$,

$$\langle t \otimes \alpha, s \otimes \beta \rangle = (t, s) \cdot \alpha \bar{\beta}.$$

Now, we can make the exterior algebra over $T^{0,1}$,

$$\Lambda^{0,*} = \Lambda^*(T^{0,1}) = \bigoplus_{p \geq 0} \Lambda^{0,p}(V \otimes \mathbb{C})$$

into Clifford module as follows. Define the representation

$$\delta : V \rightarrow \text{End}(\Lambda^{0,*}),$$

by putting

$$\delta_v : \Lambda^{0,*} \xrightarrow{v \wedge} \Lambda^* \xrightarrow{\pi} \Lambda^{0,*} \text{ for any } v \in V,$$

where $\wedge$ is the wedge product, Λ^* the exterior algebra over V , and π an obvious projection. We also define

$$\delta^* : V \rightarrow \text{End}(\Lambda^{0,*})$$

such that the operator δ_v^* is adjoint to δ_v in the Hermitian metric $\langle \cdot, \cdot \rangle$ defined above, and

$$\sigma = \delta - \delta^* : V \rightarrow \text{End}(\Lambda^{0,*}).$$

One can easily check the following properties of σ .

Proposition 1.8. *For any $v \in V$, $(\sigma_v)^2 x = -\frac{1}{2}(v, v)x$.*

Proposition 1.9. *For any $v, w \in V$ such that $(v, w) = 0$, $\sigma_v \sigma_w + \sigma_w \sigma_v = 0$.*

Therefore, we obtain a representation

$$\rho : Cl(V) \rightarrow \text{End}(\Lambda^{0,*})$$

by putting $\rho(v)(x) = \sqrt{2}\sigma_v(x)$. The (complex) dimension of this representation is equal to $\dim(\Lambda^{0,*}) = 2^m$, it must therefore be a copy of the spin representation.

One can split the algebra $\Lambda^{0,*}$ according to the parity as

$$\Lambda^{0,*} = \Lambda^{0,+} \oplus \Lambda^{0,-}, \quad \Lambda^{0,\pm} = \bigoplus_{p \text{ even/odd}} \Lambda^{0,p}. \quad (1.10)$$

On the space V , regarded as a subset of $Cl(V)$, the spin representation ρ then looks like

$$\rho(v) = \begin{pmatrix} 0 & \rho^+(v) \\ \rho^-(v) & 0 \end{pmatrix}, \quad \rho^\pm(v) : \Lambda^{0,\pm} \rightarrow \Lambda^{0,\mp}.$$

On the other hand, restricting to the even part $Cl_0(V)$ we see that ρ decomposes as a direct sum of representations in $\Lambda^{0,\pm}$ which we continue to denote $\rho^\pm$. These are just the aforementioned two representations of $Cl(W)$, for the odd dimensional space W (codimension 1 in V). To see that $\rho^\pm|_{Cl_0(V)}$ are non-isomorphic observe that the splitting (1.10) is according to ± 1 eigenspaces of the action of $i^m \eta \in Cl(V)$ (assume m is even or go to the complex form of Clifford algebra) and this is same element as $i^m \eta \in Cl(W)$ under the identification $Cl(W) \cong Cl_0(V)$. From now on, denote $S = \Lambda^{0,*}$ and $S^\pm = \Lambda^{0,\pm}$ and call representations $\rho^\pm$ the **half-spinor representations**.

Having the spin representation one can define a (local) Dirac operator

$$D : C_V^\infty(S) \rightarrow C_V^\infty(S), \quad (1.11)$$

$$D(\psi) = \sum_{k=1}^n \rho(e_k) \frac{\partial}{\partial x_k}, \quad (1.12)$$

as well as a pair of differential operators

$$D^\pm : C_V^\infty(S^\pm) \rightarrow C_V^\infty(S^\mp), \quad (1.13)$$

$$D^\pm(\psi) = \sum_{k=1}^{2m} \rho^\pm(e_k) \frac{\partial}{\partial x_k}, \quad (1.14)$$

which are also usually called Dirac operators. These are, of course, related by

$$D = \begin{pmatrix} 0 & D^+ \\ D^- & 0 \end{pmatrix},$$

and $(D^-)^* = D^+$.

In the odd dimensional case we have Dirac operators

$$D^\pm : C_W^\infty(S^\pm) \rightarrow C_W^\infty(S^\pm),$$

defined by the same formula (1.14), where k now runs from 1 to $2m - 1 = \dim W$ and e_k are understood as the generators of $Cl(W)$. Thus, in this case, $D^- = -D^+$ by the properties of $\rho^\pm$.

It should be mentioned that the Dirac operators on the adjacent odd and even dimensional spaces can be related as follows (the subscripts indicate the number of variables),

$$D_{(2m)}^+ = \frac{\partial}{\partial x_{2m}} + D_{(2m-1)}^+ : C^\infty(S^+) \rightarrow C^\infty(S^-) \stackrel{-e_{2m}}{\cong} C^\infty(S^+).$$

Here $S^\pm$ are interchangeably thought of as assigned to W_{2m-1} or V_{2m} .

Example. Let us revisit the case $n = 3$. The Clifford algebra is isomorphic to $\mathbb{H} \oplus \mathbb{H}$, where $\mathbb{H}$ is the algebra of quaternions, and the two copies of $\mathbb{H}$ here are the plus and minus eigenspace for the action of $-\eta = -e_1 e_2 e_3$. We take as the spin representation the action of $Cl(\mathbb{R}^3)$ that factors through the “plus” subalgebra. The basis for $Cl(\mathbb{R}^3)^+$ is

$$\left\{ \frac{1-\eta}{2}, \frac{e_3+e_1e_2}{2}, \frac{e_1+e_2e_3}{2}, \frac{e_2+e_3e_1}{2} \right\}$$

corresponding to $1, i, j, k, \in \mathbb{H}$. The matrices corresponding to these elements are

$$B_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, B_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad (1.15)$$

and the Dirac operator is given by

$$D = \sum_{k=1}^3 B_k \nabla_k = \begin{pmatrix} i\nabla_3 & -\nabla_1 - i\nabla_2 \\ \nabla_1 - i\nabla_2 & -i\nabla_3 \end{pmatrix}.$$

This is, of course, the same operator that we saw in the Pauli matrices example (because, as we see now, Pauli matrices generate $Cl(\mathbb{R}^3)$).

To obtain D^+ on $\mathbb{R}^4$, define A_k to be the 4×4 matrices $\begin{pmatrix} 0 & B_k \\ -B_k^* & 0 \end{pmatrix}$ with $B_k : S^+ \rightarrow S^-$, $k = 0, 1, 2, 3$ as in (1.15), and put $D^+ = \nabla_0 + D$.

DIRAC OPERATOR ON MANIFOLDS

Spin structures

The construction of a (local) Dirac operator above was a “flat space” construction, and our aim now is to generalize it to an oriented Riemannian manifold. If X is such a manifold then the tangent bundle TX has a structure of a vector $SO(n)$ -bundle. The fibres $T_x X$ are inner product spaces, so we can find an orthonormal basis $e_1, \dots, e_n$ in each fibre and form the bundle $Cl(TX)$ of Clifford algebras. The fibre at $x \in X$ of this bundle is just the Clifford algebra $Cl(T_x X)$ on generators $e_1, \dots, e_n$. At any fixed $x \in X$, there exists a spin representation $\rho_x : Cl(T_x X) \rightarrow \text{End}(S)$ (agree that it is always chosen to be the “positive” spin representation when $\dim X$ is odd), and the problem of defining a Dirac operator on X amounts to patching together these representations over the whole X . Formally speaking, we need to construct a *bundle S of Clifford modules* whose fibre S_x over $x \in X$ is a module over $Cl(T_x X)$ with respect to the spin representation $\rho_x : Cl(T_x X) \rightarrow \text{End}(S_x)$. The sections of S are to play the role of the $\mathbb{C}^N$ -valued functions in the “flat” case.

Let the system of maps $\varphi_{\alpha,\beta} : U_\alpha \cap U_\beta \rightarrow SO(n)$ be an $SO(n)$ -cocycle for TX . One can arrange it in such a way that all the intersections $U_{\alpha_1} \cap \dots \cap U_{\alpha_k}$ are contractible.

Over each U_α the bundle TX is trivial, i.e. is of the form $U_\alpha \times \mathbb{R}^n$, so the bundle of Clifford algebras is trivial as well. Respectively, a Clifford module bundle can be chosen to be $U_\alpha \times S$, the module structure being given by the spin representation $\rho : Cl(\mathbb{R}^n) \rightarrow \text{End}(S)$ regardless of a point in U_α .

As the next step, we need to patch these locally defined representations together. Let $x \in U_\alpha \cap U_\beta$, then we have two orthonormal frames, $\{e_k, k = 1, \dots, n\}$ and $\{e'_k, k = 1, \dots, n\}$, TX coming from the two trivializations of TX , over U_α and U_β . These frames are related by

$$e_k = \sum_{\ell} a_k^{\ell} e_{\ell}$$

where $(a_k^{\ell}) = \varphi_{\alpha\beta}(x)$. We also have two representations

$$\rho, \rho' : Cl(T_x X) \longrightarrow \text{End}(S)$$

defined by the two choices of orthonormal frames, so $\rho'(e'_k) = \rho(e_k)$. Therefore,

$$\begin{aligned} \rho'(e_k) &= \rho'(\sum_{\ell} a_k^{\ell} e'_{\ell}) \\ &= \sum_{\ell} a_k^{\ell} \rho'(e'_{\ell}) \\ &= \sum_{\ell} a_k^{\ell} \rho(e_{\ell}) \end{aligned} \tag{1.16}$$

On the other hand, glueing representations ρ and ρ' amounts to finding an automorphism $\psi_{\alpha\beta} : S \rightarrow S$ making the following diagram commutative for any k ,

$$\begin{array}{ccc} S & \xrightarrow{\psi_{\alpha\beta}(x)} & S \\ \rho(e_k) \downarrow & & \downarrow \rho'(e_k) \\ S & \xrightarrow{\psi_{\alpha\beta}(x)} & S \end{array} \tag{1.17}$$

that is,

$$\rho'(e_k) = \psi_{\alpha\beta}(x) \rho(e_k) \psi_{\alpha\beta}(x)^{-1}.$$

First of all, such $\psi_{\alpha\beta}(x)$ does exist because the matrices $\rho'(e_k)$ induce an irreducible representation of the Clifford algebra (this is due to the fact that $\varphi_{\alpha\beta} \in SO(n)$ preserves the relations (1.1)), with the “volume element” acting as $+1$ in the case $\dim X$ is odd, and any two such irreducible representations are conjugate.

Now, at any point $x \in U_\alpha \cap U_\beta$ there are exactly two $\psi_{\alpha\beta}(x)$, differed by sign, such that (1.17) is satisfied. If $\psi_{\alpha\beta}(x)$ and $\psi'_{\alpha\beta}(x)$ are any two endomorphisms making the diagram (1.17) commutative then, for each $v \in Cl(T_x X)$,

$$\psi_{\alpha\beta} \rho(v) \psi_{\alpha\beta}(x)^{-1} = \psi'_{\alpha\beta} \rho(v) \psi'_{\alpha\beta}(x)^{-1}$$

and $\psi_{\alpha\beta}(x)^{-1} \psi'_{\alpha\beta}(x)$ lies in the centre of Clifford algebra. This means Clifford algebra on an *even-dimensional* space, whether the dimension of X is even or odd (see Lemma 1.7 in the latter case) and, therefore, $\psi_{\alpha\beta}(x)^{-1} \psi'_{\alpha\beta}(x)$ must be equal to ± 1 .

When constructing the bundle S of Clifford modules we look for a compatible set of choices of $\psi_{\alpha\beta}$, so that they form a cocycle. Since $\varphi_{\alpha\beta} \varphi_{\beta\gamma} \varphi_{\gamma\alpha} = 1$ for any α, β, γ ,

the product $\psi_{\alpha\beta}\psi_{\beta\gamma}\psi_{\gamma\alpha}$ must be equal to 1 or -1. We are free to choose a sign of each of $\psi_{\alpha\beta}$, but, in general, one can not choose signs of $\psi_{\alpha\beta}$ to satisfy the cocycle condition for all α, β, γ simultaneously. This is due to the Lie group $SO(n)$ being non-simply connected. For any $n \geq 3$, $\pi_1(SO(n)) = \mathbb{Z}_2$, therefore, there is a non-trivial double (universal) covering. The universal covering space has a structure of (simply connected) Lie group, and this group is denoted by $\text{Spin}(n)$. The problem of choosing $\psi_{\alpha\beta}$'s is in fact just an attempt to lift the $SO(n)$ cocycle $\varphi_{\alpha\beta}$ to a $\text{Spin}(n)$ -cocycle $\psi_{\alpha\beta}$.

In the case a compatible set exists, we say that the manifold X has a **Spin-structure** (or is a **spin manifold**). It can be shown that the system of $\psi_{\alpha\beta}$'s is equivalent to the second Stiefel-Whitney class of X .

Theorem 1.18. *Let X^n be an oriented Riemannian manifold. The following conditions are equivalent:*

- (i) $w_2(X) = 0$;
- (ii) *the manifold X is spin*;
- (iii) *the tangent bundle TX has a $\text{Spin}(n)$ -structure.*

Examples. A tangent bundle of any orientable 3-manifold can be trivialized. Thus, any oriented 3-manifold is spin.

This is no longer true for the 4-manifolds. The second Stiefel-Whitney class $w_2(\mathbb{CP}^2)$ of $\mathbb{CP}^2$ is not zero, it is the generator the group $H^2(\mathbb{CP}^2; \mathbb{Z}_2) = \mathbb{Z}_2$. On the other hand, the Kummer surface $K3$ has spin structure. For any closed simply connected smooth manifold X , it is spin if and only if its intersection form Q_X is even.

Let X be a spin manifold, and $S(X)$ a bundle of Clifford modules. The **Dirac operator** on X is the first order differential operator on $C^\infty(S)$ defined by the following composition:

$$C^\infty(S) \rightarrow C^\infty(T^*X \otimes S) \rightarrow C^\infty(TX \otimes S) \rightarrow C^\infty(S)$$

where the first arrow is given by the covariant derivative associated with the lifting of the Levi-Civita connection on TX , the second by the metric on X , and the third one by the Clifford algebra representation.

In terms of local orthonormal basis $\{e_1, \dots, e_n\}$ of sections of TX , one can write

$$Ds = \sum_k \rho(e_k) \nabla_k s. \quad (1.19)$$

When $\dim X$ is even the bundle $S(X)$ splits and, similarly to the local case, the Dirac operator acts as

$$D = \begin{pmatrix} 0 & D^+ \\ D^- & 0 \end{pmatrix}, \quad D^\pm : C^\infty(S^+) \rightarrow C^\infty(S^-).$$

When $\dim X$ is odd there are two Dirac operators, one for each of the Clifford module structures.

The sections of $S^\pm$ are sometimes called the **positive** (respectively, **negative**) **spinor fields**.

Spin^c structures

One can replace the condition on X to have a Spin-structure by a weaker condition which still allows to define a Dirac operator. This new structure is called a Spin^c-structure, and it exists on a much broader class of manifolds (including all smooth 4-dimensional manifolds).

Let X again be an oriented Riemannian manifold and $w_2(X) \in H^2(X; \mathbb{Z}_2)$ its second Stiefel-Whitney class. Let $w_2(X) = [w] \in H^2(X; \mathbb{Z}_2)$ be the mod 2 reduction of an integral class $\mathcal{W} \in H^2(X; \mathbb{Z})$, and λ the corresponding complex line bundle (that is, the first Chern class $c_1(\lambda)$ is $\mathcal{W}$). Then the bundle λ admits a square root, a line bundle $\lambda^{1/2}$ such that $(\lambda^{1/2}) \otimes (\lambda^{1/2}) = \lambda$, if and only if $c_1(\lambda)$ is even. That is to say, the obstruction to finding a square root of λ is $w_2(X)$, the same class that gives the obstruction to the existence of Spin-structure on X . But this essentially means that *while we may not always be able to construct the spinor bundle and $\lambda^{1/2}$, we can still construct their tensor product.*

The cocycle description for this product is provided by the maps

$$G_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow (\text{Spin}(n) \times S^1)/\sim, \text{ where } (q, \theta) \sim (-q, -\theta).$$

The right-hand side is the twisted product $\text{Spin}(n) \times_{\mathbb{Z}_2} S^1 \stackrel{\text{def}}{=} \text{Spin}^c(n)$ and determines the structure group of the corresponding bundle. We can think of this Spin^c-bundle as

$$S^c(X) = S(X) \otimes \lambda^{1/2}$$

where $S(X)$ is the fundamental spinor bundle for the possibly non-existent Spin-structure on X (as before, we take $S(X)$ to be the bundle $S^+(X)$ of positive spinors if the dimension of X is odd), and where $\lambda^{1/2}$ is the possibly non-existent square root of λ with $c_1(\lambda) = w_2(X) \bmod 2$. The bundle λ is called the **determinant line bundle** of the Spin^c-bundle $S^c(X)$.

Like $S(X)$ above, $S^c(X)$ is a bundle of Clifford modules. However, there is no longer a canonical lifting of connections from TX to $S^c(X)$, since $\text{Spin}^c(n) \rightarrow SO(n)$ is not a finite covering. We need a $U(1)$ -connection A on the determinant line bundle λ . Then, Levi-Civita and A determine a connection on $SO(n) \times S^1$ bundle, which is the quotient of $S^c(X)$ by $\{\pm 1\} \subset \text{Spin}^c(n)$, so there is a unique lifting to a connection on $S^c(X)$.

The unitary connection induced by A on $\lambda^{1/2}$ (when $\lambda^{1/2}$ exists) can be seen as follows. In local charts $U_\alpha \cap U_\beta$ we have

$$A^{(\alpha)} = \varphi'^{-1}_{\alpha\beta} d\varphi'_{\alpha\beta} + A^{(\beta)} \tag{1.20}$$

and we know that there exist (at least locally) a cocycle map $\psi'_{\alpha\beta}$ such that $\psi'^2_{\alpha\beta} = \varphi'_{\alpha\beta}$. So, after substitution of $\psi'^2_{\alpha\beta}$, the formula (1.20) becomes

$$A'^{(\alpha)} = 2\psi'^{-1}_{\alpha\beta} d\psi'_{\alpha\beta} + A'^{(\beta)},$$

which tells us that the connection induced on $\lambda^{1/2}$ is given by the forms $\frac{1}{2}A^{(\alpha)}$ in local charts. Locally, both $S(X)$ and $\lambda^{1/2}$ exist and $S(X)$ carries a canonical (Riemannian) connection. Give the bundle $S^c(X) = S(X) \otimes \lambda^{1/2}$ the tensor product connection. It is not difficult to see that this connection is defined globally.

The Dirac operator D_A (depending on the choice of λ and a connection A on λ) is defined as follows:

$$D_A : C^\infty(S^c) \rightarrow C^\infty(S^c), \quad (1.21)$$

or, locally,

$$D_A(\psi \otimes s) = D\psi \otimes s + \psi \cdot \nabla_A(s) \quad (1.22)$$

where $\cdot$ stands for Clifford action. One can also use the formula (1.19) with $\nabla = \nabla_A$ now understood as a connection on $S^c(X)$. Similarly, D_A is a self-adjoint first order operator and the splitting of $S^c(X)$ defines two components D_A^+ and D_A^- which are also called Dirac operators.

NOTES. A general guidance for many of the topics discussed here was provided by [K]. The approach to the representation theory of the Clifford algebras comes from the even-dimensional case discussion in [R, Ch.2] (where it is referred to J.F. Adams and, further, to Eckmann). The odd-dimensional case is completed using material in [M, Ch.2]. The concept of the Dirac operator associated to a Clifford bundle follows [R, Ch.2 and 10]. A standard reference for spin-manifolds is [LM], Chapter II there discusses the relation to the Dirac operators.

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Lecture 2.

On the three-dimensional Seiberg–Witten invariant

SEIBERG–WITTEN EQUATIONS ON THREE-MANIFOLDS

Consider X a compact connected oriented Riemannian 3-manifold. Recall from the previous lecture that a bundle of Clifford algebras corresponding to Spin structure on X is a rank 2 complex Hermitian vector bundle on which the Clifford module structure can be chosen in two non-equivalent ways. The choices correspond to the two non-isomorphic irreducible representations of the Clifford algebra $Cl(\mathbb{R}^3)$. Further, recall that in dimension 3 we agreed to fix once and for all the choice of the representation which factors through $Cl^+(\mathbb{R}^3)$ and, respectively we chose to work with the positive spinor fields over 3-manifolds. With this said, we shall normally drop the $+$ subscript from the notation and write $S(X)$ for the (positive) fundamental spinor bundle over X .

Once we have fixed a Spin structure on X , it is very easy to classify the Spin^c structures: they are bijectively parametrized by the complex line bundles $\lambda \rightarrow X$, the complex spinor bundle associated to the Spin^c structure being $S^c(X) = S(X) \otimes \lambda^{1/2}$. Denote by $\mathcal{S}$ the set of all Spin^c structures on X and identify $\mathcal{S}$ with the integral cohomology group $H^2(X)$.

Some low-dimensional coincidences and auxiliary maps

Note that the following Lie group isomorphisms occur in dimensions 3 and 4:

$$\begin{aligned}\text{Spin}(3) &= SU(2), \\ \text{Spin}^c(3) &= U(2), \\ \text{Spin}(4) &= SU(2) \times SU(2), \\ \text{Spin}^c(4) &= S(U(2) \times U(2)),\end{aligned}$$

where $S(U(2) \times U(2))$ means $SU(4) \cap (U(2) \times U(2))$. In the previous lecture we saw (although it was not stated exactly the same way) that the representation of $\text{Spin}(3)$, respectively $\text{Spin}^c(3)$, in S_3 is just the fundamental representation of $SU(2)$, respectively $U(2)$, in $\mathbb{C}^2$. Here, S_3 is the spin representation space for the Clifford algebra $Cl(\mathbb{R}^3)$ and the aforementioned representation of Spin group in S_3 is the one appearing in the definition of Spin structure on a manifold.

In what follows, it will be convenient to use the composition of the ‘quantization map’ into Clifford algebra and its spin representation, denoted by

$$\mathfrak{c} : \Lambda^2(\mathbb{R}^3) \rightarrow \text{End}(S_3)$$

and generated by putting $\mathfrak{c}(e_k \wedge e_\ell) = \rho_3(e_k e_\ell)$ for any $k \neq \ell$. This map identifies $\Lambda^2 V$ with the space of traceless, skew-adjoint endomorphisms of S_3 . Further, $\mathfrak{c}$ extends by complex linearity to a map from $\Lambda^2 V \otimes \mathbb{C}$ to the space of traceless endomorphisms of S_3 , and the purely imaginary 2-forms then are mapped to self-adjoint endomorphisms.

Another important ingredient in Seiberg–Witten equations is the quadratic form

$$q : \varphi \in S_3 \rightarrow q(\varphi) = (\varphi \otimes \varphi^*) - \frac{1}{2}|\varphi|^2 \in \text{End}(S_3)$$

defined by $q(\varphi)\psi = \langle \psi, \varphi \rangle \varphi - \frac{1}{2}|\varphi|^2 \psi$, $\psi \in S_3$. Now, we have $S_3 \cong \mathbb{C}^2$ as Hermitian linear spaces. Accordingly, choosing an orthonormal basis in S_3 , the quadratic form q can be expressed as

$$\varphi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto q(\varphi) = \begin{pmatrix} \frac{1}{2}(|\alpha|^2 - |\beta|^2) & \alpha\bar{\beta} \\ \bar{\alpha}\beta & \frac{1}{2}(|\beta|^2 - |\alpha|^2) \end{pmatrix}.$$

It is not difficult to see that q is $\text{Spin}(3)$ -equivariant and, therefore, extends to a well-defined bundle map between the $\text{Spin}^c(3)$ vector bundle $S^c(X)$ and the associated bundle $\text{End } S^c(X)$.

Seiberg–Witten moduli space

Choose a Spin^c structure $s \in \mathcal{S}$ on X . Let A be a connection on the corresponding determinant line bundle $\det(S^c(X))$, F_A the curvature form of A , and $\psi \in \Gamma(S^c(X))$ a positive spinor field. Seiberg–Witten equations on X are a system of non-linear PDEs relating A and ψ ,

$$\begin{aligned} \mathfrak{c}(F_A) &= q(\psi), \\ D_A \psi &= 0. \end{aligned} \tag{2.23}$$

It proves useful at times to be able to perturb the equations (2.23). To this end, consider a fixed closed differential 2-form μ on X . Then the perturbed Seiberg–Witten equations are then written as

$$\begin{aligned} \mathfrak{c}(F_A + i\mu) &= q(\psi), \\ D_A \psi &= 0. \end{aligned} \tag{2.23'}$$

Seiberg–Witten equations are equations from gauge theory, originating from advances in Mathematical Physics. Geometrically, they are defined using connections and sections on vector bundles, the Spin^c bundles. The natural symmetry, ‘gauge symmetry’, of these equations comes from the action of the group of automorphisms of the Spin^c bundle over X which cover the identity of the frame bundle of TX . Now the bundle automorphisms of $S^c(X)$ are sections of a bundle of groups whose fibre is the center S^1 of $\text{Spin}^c(3) = U(2)$. This bundle of groups corresponds to the conjugation action of an abelian group S^1 on itself, $g \cdot h = ghg^{-1}$, and is, therefore, trivial. Thus the automorphisms of $S^c(X)$ are just the smooth maps from X to S^1 . Note that if u is an automorphism of $S^c(X)$ then the induced automorphism of the unitary line bundle $\lambda = \det S^c(X)$ is $u^2 \in C^\infty(X; S^1)$. The group $C^\infty(X; S^1)$ acts on the solutions of Seiberg–Witten equations by

$$(A, \psi) \mapsto (A + 2u du^{-1}, u\psi), \quad u \in C^\infty(X; S^1), \tag{2.24}$$

defining an equivalence relation.

The Seiberg–Witten invariant is obtained by making a suitable count of solutions of (2.23). This, in particular, means that the pairs (A, ψ) related by ‘gauge equivalence’ (2.24) are not distinguished (as indeed one cannot otherwise expect a finite

non-trivial number of solutions). However, after dividing out by gauge symmetry, the space of solutions to Seiberg–Witten equations may still degenerate or depend on auxiliary choices. A basic principle for dealing with the problem is to analyse a ‘generic’ perturbation of the equations.

Remark. Here, the term ‘generic’ refers to the situation when a family of perturbations depends on a parameter. A Baire set of perturbations is one corresponding to an intersection of countably many open dense subsets in the range of the parameter. Then a **generic perturbation** is one whose parameter is contained in some fixed Baire set.

Denote by $\mathcal{M}$ the set of equivalence classes of the triples (s, A, ψ) where s is a Spin^c structure on X , and (A, ψ) is a solution of (2.23′) for the corresponding Spin^c bundle, and the equivalence relation is given by the formula (2.24). The set $\mathcal{M}$ is called the **moduli space** of solutions for Seiberg–Witten equations. In general, $\mathcal{M}$ depends on the choice of μ ; however, we suppress this in the notation.

At this point, let us impose further conditions on the 3-manifold X . Assume that X is closed and that its first Betti number satisfies $b_1 > 1$ (the theory also works for the case $b_1 = 1$, with slightly less tidy results—we shall not discuss that here).

It is convenient to introduce a map Φ assigning to a triple (s, A, ψ) the 2-form $(2\pi i)^{-1}F_A$. By Chern–Weil theory, the De Rham cohomology class $[(2\pi i)^{-1}F_A] \in H_{DR}^2(X)$ is the image of Chern class $c_1(\lambda)$ in the natural map $H^2(X) \rightarrow H_{DR}^2(X)$. The latter map kills the torsion of $H^2(X)$, so we obtain a well-defined projection

$$\Phi : \mathcal{M} \rightarrow H^2(X)/\text{Tor}. \quad (2.25)$$

Thus Φ gives a natural partition of the moduli space. Remember that there is a one-to-one correspondence between $\mathcal{S}$ and the image of Φ .

The following theorems are proved by the methods of gauge theory.

Theorem 2.26. *For a 3-manifold X as above, and for a generic perturbation μ of the Seiberg–Witten equations on X , the moduli space $\mathcal{M}$ has a natural structure of a real 0-dimensional, compact, oriented manifold.*

Theorem 2.27. *The oriented manifold $\mathcal{M}$ provided by the previous theorem and the projection $\Phi : \mathcal{M} \rightarrow H^2(X)$ depend only on the orientation preserving diffeomorphism class of X .*

A compact 0-dimensional manifold is simply a finite collection of points and the orientation is a choice of sign ± 1 for each of the points. According to Theorem 2.27, the equations (2.23′) generically have solutions only for finitely many Spin^c structures on X .

Definition. Let X be a 3-manifold as above. Seiberg–Witten invariant of X is a map $SW : \mathcal{S} \rightarrow \mathbb{Z}$ assigning to a Spin^c structure s on X the sum of signs of all the points in $\Phi^{-1}(s)$, $s \in \mathcal{S} \cong H^2(X)/\text{Tor}$, where the signs are determined by the orientation on $\mathcal{M}$.

In summary, the discussed so far geometric constructions on an oriented Riemannian smooth 3-manifold X (satisfying certain restrictions) lead to the map SW depending only on the properties of X as a smooth manifold and expressible in terms of the topology of X . That is to say, the map SW is a *differential-topological invariant* of X .

ON THE WORK OF MENG AND TAUBES

Seiberg–Witten invariant and R -torsion

The Seiberg–Witten invariant map SW can be arranged as a single element of a certain cohomology group ring, via the following algebraic construction used by Meng and Taubes.

Fix a Spin^c structure $a \in \mathcal{S}$ on X . Recall that for any $s \in \mathcal{S}$ the difference of Chern classes for the corresponding determinant line bundles is even, $c_1(\lambda_s) - c_1(\lambda_a) = 2z$, $z \in H^2(X)/\text{Tor}$. Define the function $SW_a : H^2(X)/\text{Tor} \rightarrow \mathbb{Z}$, depending on the parameter a , by the formula

$$SW_a(z) = \sum_{s \in \mathcal{S}, c_1(\lambda_s) - c_1(\lambda_a) = 2z} SW(s).$$

It is not difficult to check that (1) the above definition is equivariant with respect to the action of $H^2(X)/\text{Tor}$, and (2) the function SW_a is finitely supported for every $a \in \mathcal{S}$. Here the action of $H^2(X)/\text{Tor}$ is additive on itself and by the pull-back on the space of the integer valued functions on $H^2(X)/\text{Tor}$.

In a more formal language, abbreviate $H^2(X)/\text{Tor}$ as H and consider the ring $\mathbb{Z}[H]$ of finitely supported functions on H . Then $\mathbb{Z}[H]$ is equivalent to the integral group ring of H and the functions SW_a define an element in $\underline{SW} \in \mathbb{Z}[H]/H$.

We remarked at the end of the previous page that the expression of Seiberg–Witten invariant uses only the topology of X . On the other hand, the algebraic topological invariant $\tau(X)$, sometimes called Milnor–Turaev torsion, is an element in $\mathbb{Z}[H]/H$ and behaves similarly to $SW(X)$, e.g. under the connected sum operation. In fact, the following result is due to Meng and Taubes.

Theorem 2.28. *Let X be a closed oriented Riemannian manifold. Assume that $b_1(X) > 1$. Then $\underline{SW}(X) = \tau(X)$.*

The boundary (non-compactness) issues

It is possible to generalize the Seiberg–Witten invariant and Theorem 2.28 to some cylindrical end 3-manifolds.

Take X to be a connected oriented complete Riemannian manifold. But X is not compact: it may have finitely many ends diffeomorphic and isometric to half-cylinders $T^2 \times \mathbb{R}_+$, where T^2 denotes a torus. Then Euler characteristic of X vanishes. Again, we require that $b_1(X) > 1$.

To simplify the notation, let us write N for the disjoint union of the cross-sections of all the cylindrical ends of X . So, N is a disjoint (assumed non-empty) union of tori and X decomposes as a union

$$X = X_0 \cup_N (\mathbb{R}_+ \times N)$$

of two components identified along their common boundary N . We note in passing that the compact manifold X_0 has the same topology as X .

The main feature of the generalization of Seiberg–Witten theory to a cylindrical end manifold concerns imposing suitable “boundary conditions at infinity”. Redefine $\mathcal{S}$ to

be the set of Spin^c structures on X for which the bundle $\det(S^c(X))$ is trivial over $\text{pt} \times N$. Then a curvature form F_A on $\det(S^c(X))$ restricts to an exact 2-form over N and $\mathcal{S}$ is bijectively parametrized by the abelian group $\text{Ker}(H^2(X) \rightarrow H^2(N))$.

Seiberg–Witten equations over X are complemented with the following “finite energy” type condition

$$\int_X |F_A|^2 d\text{vol}_X < \infty. \quad (2.29)$$

Finally, the perturbation 2-form μ is constrained over the cross-section N to be non-zero and covariant constant, and $*\mu|_N = \nu|_N$ for some 1-form ν defined and closed on X .

Respectively, the Seiberg–Witten moduli space $\mathcal{M}$ over a cylindrical end manifold X is defined as a set of all the equivalence classes of solutions of (2.23) and (2.29).

With these modifications imposed, the assertions of Theorems 2.26 and 2.27 hold for $\mathcal{M}$, with one exception, regarding the compactness. In fact, the map Φ has to be recast as follows.

As was remarked, the curvature F_A of any connection A on a determinant line bundle coming from $\mathcal{S}$ restricts to an exact 2-form on the ends $\mathbb{R}_+ \times N$. It can be further shown by the analytic arguments that the condition (2.29) implies the exponential decay of the form F_A over the ends and that $(2\pi i)^{-1}F_A$ represents a well-defined map on the relative homology $H_2(X, N) \rightarrow \mathbb{Z}$ with integer coefficients. The latter map is identified with an element of the torsion-free part of integral cohomology group with compact support $H_c^2(X)/\text{Tor}$. The map Φ as in (2.25) can now be defined via the natural inclusion $H_c^2(X)/\text{Tor} \subset H^2(X)/\text{Tor}$. Although the image of Φ is not identified with $\mathcal{S}$ in this case, there is a commutative diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{[(2\pi i)^{-1}F_A]} & H_c^2(X)/\text{Tor} \\ c_1 \downarrow & & \downarrow \\ H^2(X) & \longrightarrow & H^2(X)/\text{Tor}, \end{array}$$

where c_1 denotes the operation of taking the first Chern class of the determinant line bundle for $s \in \mathcal{S}$, of $[(s, A, \Phi)] \in \mathcal{M}$.

Now, even though the “global” moduli space $\mathcal{M}$ is not generally compact, it decomposes into compact pieces defined as the inverse images of points under Φ . And this is enough for the map SW to be well defined.

The invariant SW defines, with the help of the above commutative diagram, an element $\underline{SW} \in \mathbb{Z}(H)/H$. The following result is again due to Meng and Taubes (a choice of homology orientation for X is assumed).

Theorem 2.30. *Let X be a cylindrical end 3-manifold with $b_1(X) > 1$ and such that the cross-section of any cylindrical end is the torus. Then the Seiberg–Witten invariant $\underline{SW}$ equals the Milnor–Turaev torsion $\tau(X)$.*

NOTES. The descriptions of the ‘quantization map’ and quadratic form appearing in Seiberg–Witten equations are taken from [N]. The results on the Seiberg–Witten moduli space given here follow the

corresponding theorems in 4-dimensional theory [M], in conjunction with the remark in [MT] on the reduction to a 3-manifold by considering $S^1 \times X^3$. See also [MST] for a discussion of 3-dimensional Seiberg–Witten theory. The equivalence between 3-dimensional Seiberg–Witten invariant and Milnor–Turaev torsion is announced in [MT], see also [T] where an account of the torsion invariant can be found.

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