Lecture 1.
Seiberg–Witten theory: preliminaries

DIRAC OPERATORS (the local theory)

Factorizations of the Laplace operator

Let $C^\infty(U;\mathbb{C}^N)$ be the linear space of the smooth $\mathbb{C}^N$-valued functions on an open set $U \subset \mathbb{R}^n$, and

$$\Delta : C^\infty(U;\mathbb{C}^N) \longrightarrow C^\infty(U;\mathbb{C}^N), \quad \Delta = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2},$$

the Laplace operator. Any first order differential operator

$$D : C^\infty(U;\mathbb{C}^N) \longrightarrow C^\infty(U;\mathbb{C}^N)$$

such that $D^2 = \Delta$ is called a (local) **Dirac operator**.

Let $D = \sum_{k=1}^n A_k \frac{\partial}{\partial x_k}$ where A_k are complex $(N \times N)$–matrices. Then

$$\begin{aligned} D^2 &= \sum_{k,\ell} A_k \frac{\partial}{\partial x_k}(A_\ell \frac{\partial}{\partial x_\ell}) \\ &= \sum_{k,\ell} A_k A_\ell \frac{\partial^2}{\partial x_k \partial x_\ell} \\ &= \sum_{k} A_k^2 \frac{\partial^2}{\partial x_k^2} + \sum_{k<\ell} (A_k A_\ell + A_\ell A_k) \frac{\partial^2}{\partial x_k \partial x_\ell}. \end{aligned}$$

and one can easily see that the condition $D^2 = \Delta$ implies that

$$\begin{cases} A_k^2 = -1 \ \text{ for all } k, \text{ and} \\ A_k A_\ell + A_\ell A_k = 0 \ \text{ for all } k \neq \ell \end{cases} \tag{1.1}$$

As soon as we have n matrices $A_1, \ldots, A_n$ satisfying the conditions (1.1), we have a Dirac operator.

Examples

n=1. Take $N = 1$ and $A_1 = i$, then $D = i\frac{\partial}{\partial x}$ is a Dirac operator.

n=2. Let $N = 2$ and A_1 and A_2 are the matrices

$$A_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

n=3 ('Pauli spin matrices') One can take $N = 2$ again, then A_1 and A_2 can be chosen as above, and

$$A_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

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To generalize these simple examples to arbitrary dimensions we need to take a closer look at the nature of the matrices $A_1, \dots, A_n$.

Representation theory of Clifford algebras

Let $e_1, \dots, e_n$ be an orthonormal basis in $\mathbb{R}^n$. **Clifford algebra** $Cl(\mathbb{R}^n)$ is the $\mathbb{R}$–algebra generated by $e_1, \dots, e_n$ with the relations $e_k^2 = -1$ for any $k \leq n$, and $e_k e_\ell + e_\ell e_k = 0$ for any $k \neq \ell$. Any $\mathbb{C}$–linear representation

$$\rho : Cl(\mathbb{R}^n) \longrightarrow \mathrm{End}(S)$$

of this algebra yields a set of matrices $\rho(e_k),\ k = 1, \dots, n$, satisfying the relations (1.1) (the complex linear space S plays the role of $\mathbb{C}^N$ above), and vice versa, any set $A_1, \dots, A_n$ of $(N \times N)$–matrices satisfying (1.1) uniquely determines a representation ρ by putting $S = \mathbb{C}^N$ and $\rho(e_k) = A_k$. So, we want to understand the representation theory of $Cl(\mathbb{R}^n)$. It turns out that our conclusions will depend on whether n is odd or even (as a matter of fact, we are mostly interested in $n = 3$ and $n = 4$).

We shall analyse the representation theory of $Cl(\mathbb{R}^n)$ by looking at the representation theory of a finite multiplicative subgroup. Let E be the group of order 2^{n+1} consisting of all the elements

$$\pm e_1^{\varepsilon_1} e_2^{\varepsilon_2} \cdots e_n^{\varepsilon_n}$$

where each of $\varepsilon_1, \dots, \varepsilon_n$ is either 0 or 1 (check that this is indeed a group). In particular, E contains $(-1) \in \mathbb{R}$; denote this by ν when it is considered as an element of E.

Proposition 1.2. *There is a one-to-one correspondence between representations of $Cl(\mathbb{R}^n)$ and those of E on which ν acts as $-I$.*

Proof. Obvious. The condition on ν comes from the fact that for any linear representation ρ of $Cl(\mathbb{R}^n)$, we must have $\rho(\nu) = \rho(e_k^2) = (\rho(e_k))^2 = -I$. □

Thus, one can use the representation theory of the finite group E to study that of $Cl(\mathbb{R}^n)$. We need the following three theorems about representations of finite groups.

Theorem 1.3. *All irreducible representations of a finite abelian group are 1-dimensional, and their number equals the order of the group.*

Theorem 1.4. *The number of irreducible representations of a finite group is equal to the number of the conjugacy classes in the group.*

Theorem 1.5. *Let G be a finite group, then the sum of squares of dimensions of its irreducible representations equals the order of the group.*

Now, the centre of the group E consists of 1, ν, $\eta = e_1 \cdots e_n$, and $\nu\eta$ in the case of odd n, and only of 1 and ν when n is even. It can be seen as follows. Let $g = e_1^{\varepsilon_1} \cdots e_n^{\varepsilon_n}$ be central. Suppose that there exist r and s such that $\varepsilon_r = 1$, $\varepsilon_s = 0$, then one can check by hand that $e_r e_s g = \nu g e_r e_s$, and we get a contradiction. Therefore, all ε_k's should be simultaneously 0 or 1, and the only possible central elements are 1, ν, $\eta = e_1 \cdots e_n$, and $\nu\eta$. But if $n = 2m$ is even we have

$$e_1 \eta = \nu \eta e_1,$$

so the claim follows.

Since ν is a central involution in the finite group E, we must have $\rho(\nu) = \pm 1$ for any irreducible representation ρ of E. Those irreducible representations ρ for which $\rho(\nu) = 1$ are representations of the abelian group $E/\{1, \nu\}$ of order 2^n, so according to Theorem 1.3 there are 2^n of them. How many more representations does E have?

We begin with the case of even n. Let us count the conjugacy classes in E. The conjugacy class of $g \in E$ must be either $\{g\}$ (if g is central) or $\{g, \nu g\}$ (otherwise). This is an easy consequence of the fact that $E/\{1, \nu\}$ is abelian. The number of conjugacy classes is therefore

$$\frac{2^{n+1} - 2}{2} + 2 = 2^n + 1.$$

Due to Theorem 1.4, so is the number of irreducible representations of E. It follows that E has, up to conjugation, just one more irreducible representation ρ, and $\rho(\nu)$ must be -1. The corresponding representation of the Clifford algebra is called the **spin representation**. Its dimension can be calculated by using Theorem 1.5. Namely,

$$2^n + (\dim \rho)^2 = 2^{n+1}.$$

Therefore, $\dim \rho = 2^{\,n/2}$ (remember that n is even).

Now assume that $n \equiv 3 \mod 4$. A simple computation shows that $\eta^2 = 1$ in this case and that η is independent of the choice of oriented orthogonal basis. The left multiplication by $-\eta$ gives a canonical decomposition of $Cl(\mathbb{R}^n)$ into the ± 1 eigenspaces

$$Cl(\mathbb{R}^n) = Cl(\mathbb{R}^n)^+ \oplus Cl(\mathbb{R}^n)^-. \tag{1.6}$$

Since $-\eta$ is contained in the centre of $Cl(\mathbb{R}^n)$, this means that $Cl(\mathbb{R}^n)^\pm$ are subalgebras which annihilate each other and (1.6) is an orthogonal sum of algebras.

Lemma 1.7. *If $n \equiv 3 \mod 4$ then the algebras $Cl(\mathbb{R}^n)^\pm$ are both isomorphic to $Cl(\mathbb{R}^{n-1})$.*

Proof. The required isomorphisms of algebras are obtained as the compositions

$$Cl(\mathbb{R}^{n-1}) \xrightarrow{[e_n]} Cl(\mathbb{R}^n) \xrightarrow{\pi^\pm} Cl(\mathbb{R}^n)^\pm,$$

where the embedding $[e_n]$ is generated by putting $e_k \mapsto e_n e_k$ for each k, with $e_1, \dots, e_{n-1}$ being an orthonormal basis of $\mathbb{R}^{n-1}$, and $\pi^\pm$ are the obvious projections. The maps $\pi^\pm$ are expressed as left multiplications by $(1 \mp \eta)/2$ and the lemma follows as the image of $[e_n]$ in $Cl(\mathbb{R}^n)$ meets both $Cl(\mathbb{R}^n)^\pm$ trivially. □

Notice that the embedding map $[e_n]$ is the graph of an algebra isomorphism $Cl(\mathbb{R}^n)^+ \to Cl(\mathbb{R}^n)^-$.

Remark. The decomposition (1.6) and Lemma 1.7 generalize to any odd n if one considers the complex form $Cl(\mathbb{R}^n) \otimes \mathbb{C}$ of Clifford algebra and replaces the element $-\eta$ with $i^{[\frac{n+1}{2}]} e_1 \cdots e_n$.

The subalgebra of $Cl(\mathbb{R}^n)$ obtained as the image of $[e_n]$ makes sense for any n. This is the **even part of Clifford algebra**, denoted $Cl_0(\mathbb{R}^n)$, defined as a linear span of

the basis elements $e_{j_1}\cdots e_{j_k}$, where k is even. Respectively, by taking k odd one defines the **odd part** $Cl_1(\mathbb{R}^n)$ of $Cl(\mathbb{R}^n)$ and there is a $\mathbb{Z}_2$-grading

$$Cl(\mathbb{R}^n) = Cl_0(\mathbb{R}^n) \oplus Cl_1(\mathbb{R}^n),$$

and $Cl_0(\mathbb{R}^n) \cong Cl(\mathbb{R}^{n-1})$.

In the case of odd n, the number of conjugacy classes in E is $(2^{n+1}-4)/2+4 = 2^n+2$, so $Cl(\mathbb{R}^n)$ then has exactly two non-isomorphic irreducible, finite dimensional, complex representations up to isomorphism. If n is congruent to 3 modulo 4, they are obtained by projecting onto one of the two factors in (1.6) and taking the spin representation of that factor. Since $Cl(\mathbb{R}^{n-1})$ in this case is embedded diagonally in the direct sum, it is clear that the restrictions of the non-isomorphic irreducible complex representations to $Cl_0(\mathbb{R}^n) \cong Cl(\mathbb{R}^{n-1})$ become isomorphic. In particular, any of these two representations of $Cl(\mathbb{R}^n)$ has dimension $2^{\frac{n-1}{2}}$ (which can also be verified by elementary number theory methods, considering the constraint provided by Theorem 1.5). Note that whereas the two representations coincide on the even part $Cl_0(\mathbb{R}^n)$ they have opposite signs on the odd part $Cl_1(\mathbb{R}^n)$, in particular, on the generators $e_1, \ldots, e_n$.

The spin representation $\rho : Cl(\mathbb{R}^{2m}) \to \mathrm{End}(S)$ can be explicitly constructed as follows. Denote by V the real $2m$-dimensional Euclidean vector space, one can always assume that there exists a linear operator $J : V \to V$ such that $J^2 = -I$ compatible with the Euclidean structure, $(Jx, Jy) = (x, y)$. Then $V \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$ where

$$T^{1,0} = \{t \in V \otimes \mathbb{C} \mid Jt = it\}$$
$$T^{0,1} = \{t \in V \otimes \mathbb{C} \mid Jt = -it\}$$

are the $\pm i$ eigenspaces of J acting on $V \otimes \mathbb{C}$ as $J \otimes I$. We also define a Hermitian metric on $V \otimes \mathbb{C}$,

$$\langle t \otimes \alpha, s \otimes \beta \rangle = (t, s) \cdot \alpha\bar{\beta}.$$

Now, we can make the exterior algebra over $T^{0,1}$,

$$\Lambda^{0,*} = \Lambda^*(T^{0,1}) = \bigoplus_{p \geq 0} \Lambda^{0,p}(V \otimes \mathbb{C})$$

into Clifford module as follows. Define the representation

$$\delta : V \to \mathrm{End}(\Lambda^{0,*}),$$

by putting

$$\delta_v : \Lambda^{0,*} \xrightarrow{v\wedge\cdot} \Lambda^* \xrightarrow{\pi} \Lambda^{0,*} \text{ for any } v \in V,$$

where $\wedge$ is the wedge product, Λ^* the exterior algebra over V, and π an obvious projection. We also define

$$\delta^* : V \to \mathrm{End}(\Lambda^{0,*})$$

such that the operator δ_v^* is adjoint to δ_v in the Hermitian metric $< \cdot, \cdot >$ defined above, and

$$\sigma = \delta - \delta^* : V \to \mathrm{End}(\Lambda^{0,*}).$$

One can easily check the following properties of σ.

Proposition 1.8. *For any $v \in V$, $(\sigma_v)^2 x = -\frac{1}{2}(v, v)x$.*

Proposition 1.9. *For any $v, w \in V$ such that $(v, w) = 0$, $\sigma_v\sigma_w + \sigma_w\sigma_v = 0$.*

Therefore, we obtain a representation

$$\rho : Cl(V) \to \mathrm{End}(\Lambda^{0,*})$$

by putting $\rho(v)(x) = \sqrt{2}\sigma_v(x)$. The (complex) dimension of this representation is equal to $\dim(\Lambda^{0,*}) = 2^m$, it must therefore be a copy of the spin representation.

One can split the algebra $\Lambda^{0,*}$ according to the parity as

$$\Lambda^{0,*} = \Lambda^{0,+} \oplus \Lambda^{0,-}, \quad \Lambda^{0,\pm} = \bigoplus_{p \text{ even/odd}} \Lambda^{0,p}. \tag{1.10}$$

On the space V, regarded as a subset of $Cl(V)$, the spin representation ρ then looks like

$$\rho(v) = \begin{pmatrix} 0 & \rho^+(v) \\ \rho^-(v) & 0 \end{pmatrix}, \quad \rho^\pm(v) : \Lambda^{0,\pm} \to \Lambda^{0,\mp}.$$

On the other hand, restricting to the even part $Cl_0(V)$ we see that ρ decomposes as a direct sum of representations in $\Lambda^{0,\pm}$ which we continue to denote $\rho^\pm$. These are just the aforementioned two representations of $Cl(W)$, for the odd dimensional space W (codimension 1 in V). To see that $\rho^\pm|Cl_0(V)$ are non-isomorphic observe that the splitting (1.10) is according to ± 1 eigenspaces of the action of $i^m\eta \in Cl(V)$ (assume m is even or go to the complex form of Clifford algebra) and this is same element as $i^m\eta \in Cl(W)$ under the identification $Cl(W) \cong Cl_0(V)$. ¿From now on, denote $S = \Lambda^{0,*}$ and $S^\pm = \Lambda^{0,\pm}$ and call representations $\rho^\pm$ the **half-spinor representations**.

Having the spin representation one can define a (local) Dirac operator

$$D : C^\infty_V(S) \to C^\infty_V(S), \tag{1.11}$$

$$D(\psi) = \sum_{k=1}^{n} \rho(e_k)\, \frac{\partial}{\partial x_k}, \tag{1.12}$$

as well as a pair of differential operators

$$D^\pm : C^\infty_V(S^\pm) \to C^\infty_V(S^\mp), \tag{1.13}$$

$$D^\pm(\psi) = \sum_{k=1}^{2m} \rho^\pm(e_k)\, \frac{\partial}{\partial x_k}, \tag{1.14}$$

which are also usually called Dirac operators. These are, of course, related by

$$D = \begin{pmatrix} 0 & D^+ \\ D^- & 0 \end{pmatrix},$$

and $(D^-)^* = D^+$.

In the odd dimensional case we have Dirac operators

$$D^\pm : C^\infty_W(S^\pm) \to C^\infty_W(S^\pm),$$

defined by the same formula (1.14), where k now runs from 1 to $2m - 1 = \dim W$ and e_k are understood as the generators of $Cl(W)$. Thus, in this case, $D^- = -D^+$ by the properties of $\rho^\pm$.

It should be mentioned that the Dirac operators on the adjacent odd and even dimensional spaces can be related as follows (the subscripts indicate the number of variables),

$$D^+_{(2m)} = \frac{\partial}{\partial x_{2m}} + D^+_{(2m-1)} : C^\infty(S^+) \to C^\infty(S^-) \overset{-e_{2m}}{\cong} C^\infty(S^+).$$

Here $S^\pm$ are interchangeably thought of as assigned to W_{2m-1} or V_{2m}.

Example. Let us revisit the case $n = 3$. The Clifford algebra is isomorphic to $\mathbb{H} \oplus \mathbb{H}$, where $\mathbb{H}$ is the algebra of quaternions, and the two copies of $\mathbb{H}$ here are the plus and minus eigenspace for the action of $-\eta = -e_1e_2e_3$. We take as the spin representation the action of $Cl(\mathbb{R}^3)$ that factors through the "plus" subalgebra. The basis for $Cl(\mathbb{R}^3)^+$ is

$$\left\{ \frac{1-\eta}{2} . \frac{e_3 + e_1e_2}{2}, \frac{e_1 + e_2e_3}{2}, \frac{e_2 + e_3e_1}{2} \right\}$$

corresponding to $1, i, j, k, \in \mathbb{H}$. The matrices corresponding to these elements are

$$B_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, B_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \tag{1.15}$$

and the Dirac operator is given by

$$D = \sum_{k=1}^{3} B_k \nabla_k = \begin{pmatrix} i\nabla_3 & -\nabla_1 - i\nabla_2 \\ \nabla_1 - i\nabla_2 & -i\nabla_3 \end{pmatrix}.$$

This is, of course, the same operator that we saw in the Pauli matrices example (because, as we see now, Pauli matrices generate $Cl(\mathbb{R}^3)$).

To obtain D^+ on $\mathbb{R}^4$, define A_k to be the 4×4 matrices $\left(\begin{smallmatrix} 0 & B_k \\ -B_k^* & 0 \end{smallmatrix}\right)$ with $B_k : S^+ \to S^-$, $k = 0, 1, 2, 3$ as in (1.15), and put $D^+ = \nabla_0 + D$.

DIRAC OPERATOR ON MANIFOLDS

Spin structures

The construction of a (local) Dirac operator above was a "flat space" construction, and our aim now is to generalize it to an oriented Riemannian manifold. If X is such a manifold then the tangent bundle TX has a structure of a vector $SO(n)$-bundle. The fibres T_xX are inner product spaces, so we can find an orthonormal basis $e_1, \ldots, e_n$ in each fibre and form the bundle $Cl(TX)$ of Clifford algebras. The fibre at $x \in X$ of this bundle is just the Clifford algebra $Cl(T_xX)$ on generators $e_1, \ldots, e_n$. At any fixed $x \in X$, there exists a spin representation $\rho_x : Cl(T_xX) \longrightarrow \mathrm{End}(S)$ (agree that it is always chosen to be the "positive" spin representation when $\dim X$ is odd), and the problem of defining a Dirac operator on X amounts to patching together these representations over the whole X. Formally speaking, we need to construct a *bundle S of Clifford modules* whose fibre S_x over $x \in X$ is a module over $Cl(T_xX)$ with respect to the spin representation $\rho_x : Cl(T_xX) \longrightarrow \mathrm{End}(S_x)$. The sections of S are to play the role of the $\mathbb{C}^N$–valued functions in the "flat" case.

Let the system of maps $\varphi_{\alpha,\beta} : U_\alpha \cap U_\beta \to SO(n)$ be an $SO(n)$–cocycle for TX. One can arrange it in such a way that all the intersections $U_{\alpha_1} \cap \ldots \cap U_{\alpha_k}$ are contractible.

Over each U_α the bundle TX is trivial, i.e. is of the form $U_\alpha \times \mathbb{R}^n$, so the bundle of Clifford algebras is trivial as well. Respectively, a Clifford module bundle can be chosen to be $U_\alpha \times S$, the module structure being given by the spin representation $\rho : Cl(\mathbb{R}^n) \to \mathrm{End}(S)$ regardless of a point in U_α.

As the next step, we need to patch these locally defined representations together. Let $x \in U_\alpha \cap U_\beta$, then we have two orthonormal frames, $\{e_k,\ k = 1, \dots, n\}$ and $\{e'_k,\ k = 1, \dots, n\}$, TX coming from the two trivializations of TX, over U_α and U_β. These frames are related by

$$e_k = \sum_\ell a_k^\ell e_\ell$$

where $(a_k^\ell) = \varphi_{\alpha\beta}(x)$. We also have two representations

$$\rho,\ \rho' : Cl(T_xX) \longrightarrow \mathrm{End}(S)$$

defined by the two choices of orthonormal frames, so $\rho'(e'_k) = \rho(e_k)$. Therefore,

$$\begin{aligned} \rho'(e_k) &= \rho'(\sum_\ell a_k^\ell e'_\ell) \\ &= \sum_\ell a_k^\ell \rho'(e'_\ell) \\ &= \sum_\ell a_k^\ell \rho(e_\ell) \end{aligned} \tag{1.16}$$

On the other hand, glueing representations ρ and ρ' amounts to finding an automorphism $\psi_{\alpha\beta} : S \to S$ making the following diagram commutative for any k,

$$\begin{array}{ccc} S & \xrightarrow{\psi_{\alpha\beta}(x)} & S \\ {\scriptstyle \rho(e_k)}\downarrow & & \downarrow{\scriptstyle \rho'(e_k)} \\ S & \xrightarrow{\psi_{\alpha\beta}(x)} & S \end{array} \tag{1.17}$$

that is,

$$\rho'(e_k) = \psi_{\alpha\beta}(x)\rho(e_k)\psi_{\alpha\beta}(x)^{-1}.$$

First of all, such $\psi_{\alpha\beta}(x)$ does exist because the matrices $\rho'(e_k)$ induce an irreducible representation of the Clifford algebra (this is due to the fact that $\varphi_{\alpha\beta} \in SO(n)$ preserves the relations (1.1)), with the "volume element" acting as $+1$ in the case $\dim X$ is odd, and any two such irreducible representations are conjugate.

Now, at any point $x \in U_\alpha \cap U_\beta$ there are exactly two $\psi_{\alpha\beta}(x)$, differed by sign, such that (1.17) is satisfied. If $\psi_{\alpha\beta}(x)$ and $\psi'_{\alpha\beta}(x)$ are any two endomorphisms making the diagram (1.17) commutative then, for each $v \in Cl(T_xX)$,

$$\psi_{\alpha\beta}\ \rho(v)\psi_{\alpha\beta}(x)^{-1} = \psi'_{\alpha\beta}\ \rho(v)\psi'_{\alpha\beta}(x)^{-1}$$

and $\psi_{\alpha\beta}(x)^{-1}\psi'_{\alpha\beta}(x)$ lies in the centre of Clifford algebra. This means Clifford algebra on an *even-dimensional* space, whether the dimension of X is even or odd (see Lemma 1.7 in the latter case) and, therefore, $\psi_{\alpha\beta}(x)^{-1}\psi'_{\alpha\beta}(x)$ must be equal to ± 1.

When constructing the bundle S of Clifford modules we look for a compatible set of choices of $\psi_{\alpha\beta}$, so that they form a cocycle. Since $\varphi_{\alpha\beta}\varphi_{\beta\gamma}\varphi_{\gamma\alpha} = 1$ for any α, β, γ,

the product $\psi_{\alpha\beta}\psi_{\beta\gamma}\psi_{\gamma\alpha}$ must be equal to 1 or -1. We are free to choose a sign of each of $\psi_{\alpha\beta}$, but, in general, one can not choose signs of $\psi_{\alpha\beta}$ to satisfy the cocycle condition for all α, β, γ simultaneously. This is due to the Lie group $SO(n)$ being non-simply connected. For any $n \geq 3$, $\pi_1(SO(n)) = \mathbb{Z}_2$, therefore, there is a non-trivial double (universal) covering. The universal covering space has a structure of (simply connected) Lie group, and this group is denoted by $\mathrm{Spin}(n)$. The problem of choosing $\psi_{\alpha\beta}$'s is in fact just an attempt to lift the $SO(n)$ cocycle $\varphi_{\alpha\beta}$ to a $\mathrm{Spin}(n)$–cocycle $\psi_{\alpha\beta}$.

In the case a compatible set exists, we say that the manifold X has a **Spin-structure** (or is a **spin manifold**). It can be shown that the system of $\psi_{\alpha\beta}$'s is equivalent to the second Stiefel–Whitney class of X.

Theorem 1.18. *Let X^n be an oriented Riemannian manifold. The following conditions are equivalent:*

(i) $w_2(X) = 0$*;*
(ii) *the manifold X is spin;*
(iii) *the tangent bundle TX is has a* $\mathrm{Spin}(n)$*-structure.*

Examples. A tangent bundle of any orientable 3-manifold can be trivialized. Thus, any oriented 3-manifold is spin.

This is no longer true for the 4-manifolds. The second Stiefel-Whitney class $w_2(\mathbb{C}P^2)$ of $\mathbb{C}P^2$ is not zero, it is the generator the group $H^2(\mathbb{C}P^2; \mathbb{Z}_2) = \mathbb{Z}_2$. On the other hand, the Kummer surface $K3$ has spin structure. For any closed simply connected smooth manifold X, it is spin if and only if its intersection form Q_X is even.

Let X be a spin manifold, and $S(X)$ a bundle of Clifford modules. The **Dirac operator** on X is the first order differential operator on $C^\infty(S)$ defined by the following composition:

$$C^\infty(S) \to C^\infty(T^*X \otimes S) \to C^\infty(TX \otimes S) \to C^\infty(S)$$

where the first arrow is given by the covariant derivative associated with the lifting of the Levi-Civita connection on TX, the second by the metric on X, and the third one by the Clifford algebra representation.

In terms of local orthonormal basis $\{e_1, \ldots, e_n\}$ of sections of TX, one can write

$$Ds = \sum_k \rho(e_k)\nabla_k s. \tag{1.19}$$

When $\dim X$ is even the bundle $S(X)$ splits and, similarly to the local case, the Dirac operator acts as

$$D = \begin{pmatrix} 0 & D^+ \\ D^- & 0 \end{pmatrix}, \quad D^\pm : C^\infty(S^+) \to C^\infty(S^-).$$

When $\dim X$ is odd there are two Dirac operators, one for each of the Clifford module structures.

The sections of $S^\pm$ are sometimes called the **positive** (respectively, **negative**) **spinor fields**.

Spin^c structures

One can replace the condition on X to have a Spin-structure by a weaker condition which still allows to define a Dirac operator. This new structure is called a Spin^c-structure, and it exists on a much broader class of manifolds (including all smooth 4-dimensional manifolds).

Let X again be an oriented Riemannian manifold and $w_2(X) \in H^2(X;\mathbb{Z}_2)$ its second Stiefel-Whitney class. Let $w_2(X) = [w] \in H^2(X;\mathbb{Z}_2)$ be the mod 2 reduction of an integral class $\mathcal{W} \in H^2(X;\mathbb{Z})$, and λ the corresponding complex line bundle (that is, the first Chern class $c_1(\lambda)$ is $\mathcal{W}$). Then the bundle λ admits a square root, a line bundle $\lambda^{1/2}$ such that $(\lambda^{1/2}) \otimes (\lambda^{1/2}) = \lambda$, if and only if $c_1(\lambda)$ is even. That is to say, the obstruction to finding a square root of λ is $w_2(X)$, the same class that gives the obstruction to the existence of Spin-structure on X. But this essentially means that *while we may not always be able to construct the spinor bundle and $\lambda^{1/2}$, we can still construct their tensor product.*

The cocycle description for this product is provided by the maps

$$G_{\alpha\beta} : U_\alpha \cap U_\beta \to (\mathrm{Spin}(n) \times S^1)/\sim \text{ , where } (q,\theta) \sim (-q,-\theta).$$

The right-hand side is the twisted product $\mathrm{Spin}(n) \times_{\mathbb{Z}_2} S^1 \stackrel{\mathrm{def}}{=} \mathrm{Spin}^c(n)$ and determines the structure group of the corresponding bundle. We can think of this Spin^c-bundle as

$$S^c(X) = S(X) \otimes \lambda^{1/2}$$

where $S(X)$ is the fundamental spinor bundle for the possibly non-existent Spin-structure on X (as before, we take $S(X)$ to be the bundle $S^+(X)$ of positive spinors if the dimension of X is odd), and where $\lambda^{1/2}$ is the possibly non-existent square root of λ with $c_1(\lambda) = w_2(X) \bmod 2$. The bundle λ is called the **determinant line bundle** of the Spin^c-bundle $S^c(X)$.

Like $S(X)$ above, $S^c(X)$ is a bundle of Clifford modules. However, there is no longer a canonical lifting of connections from TX to $S^c(X)$, since $\mathrm{Spin}^c(n) \to SO(n)$ is not a finite covering. We need a $U(1)$-connection A on the determinant line bundle λ. Then, Levi–Civita and A determine a connection on $SO(n) \times S^1$ bundle, which is the quotient of $S^c(X)$ by $\{\pm 1\} \subset \mathrm{Spin}^c(n)$, so there is a unique lifting to a connection on $S^c(X)$.

The unitary connection induced by A on $\lambda^{1/2}$ (when $\lambda^{1/2}$ exists) can be seen as follows. In local charts $U_\alpha \cap U_\beta$ we have

$$A^{(\alpha)} = {\varphi'}^{-1}_{\alpha\beta} d{\varphi'}_{\alpha\beta} + A^{(\beta)} \tag{1.20}$$

and we know that there exist (at least locally) a cocycle map $\psi'_{\alpha\beta}$ such that ${\psi'}^2_{\alpha\beta} = \varphi'_{\alpha\beta}$. So, after substitution of ${\psi'}^2_{\alpha\beta}$, the formula (1.20) becomes

$$A'^{(\alpha)} = 2{\psi'}^{-1}_{\alpha\beta} d{\psi'}_{\alpha\beta} + A'^{(\beta)},$$

which tells us that the connection induced on $\lambda^{1/2}$ is given by the forms $\frac{1}{2}A^{(\alpha)}$ in local charts. Locally, both $S(X)$ and $\lambda^{1/2}$ exist and $S(X)$ carries a canonical (Riemannian) connection. Give the bundle $S^c(X) = S(X) \otimes \lambda^{1/2}$ the tensor product connection. It is not difficult to see that this connection is defined globally.

The Dirac operator D_A (depending on the choice of λ and a connection A on λ) is defined as follows:

$$D_A : C^\infty(S^c) \to C^\infty(S^c), \tag{1.21}$$

or, locally,

$$D_A(\psi \otimes s) = D\psi \otimes s + \psi \cdot \nabla_A(s) \tag{1.22}$$

where $\cdot$ stands for Clifford action. One can also use the formula (1.19) with $\nabla = \nabla_A$ now understood as a connection on $S^c(X)$. Similarly, D_A is a self-adjoint first order operator and the splitting of $S^c(X)$ defines two components D_A^+ and D_A^- which are also called Dirac operators.

NOTES. A general guidance for many of the topics discussed here was provided by [K]. The approach to the representation theory of the Clifford algebras comes from the even-dimensional case discussion in [R, Ch.2] (where it is referred to J.F. Adams and, further, to Eckmann). The odd-dimensional case is completed using material in [M, Ch.2]. The concept of the Dirac operator associated to a Clifford bundle follows [R, Ch.2 and 10]. A standard reference for spin-manifolds is [LM], Chapter II there discusses the relation to the Dirac operators.

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Lecture 2.
On the three-dimensional Seiberg–Witten invariant

SEIBERG–WITTEN EQUATIONS ON THREE-MANIFOLDS

Consider X a compact connected oriented Riemannian 3-manifold. Recall from the previous lecture that a bundle of Clifford algebras corresponding to Spin structure on X is a rank 2 complex Hermitian vector bundle on which the Clifford module structure can be chosen in two non-equivalent ways. The choices correspond to the two non-isomorphic irreducible representations of the Clifford algebra $Cl(\mathbb{R}^3)$. Further, recall that in dimension 3 we agreed to fix once and for all the choice of the representation which factors through $Cl^+(\mathbb{R}^3)$ and, respectively we chose to work with the positive spinor fields over 3-manifolds. With this said, we shall normally drop the + subscript from the notation and write $S(X)$ for the (positive) fundamental spinor bundle over X.

Once we have fixed a Spin structure on X, it is very easy to classify the Spin^c structures: they are bijectively parametrized by the complex line bundles $\lambda \to X$, the complex spinor bundle associated to the Spin^c structure being $S^c(X) = S(X) \otimes \lambda^{1/2}$. Denote by $\mathcal{S}$ the set of all Spin^c structures on X and identify $\mathcal{S}$ with the integral cohomology group $H^2(X)$.

Some low-dimensional coincidences and auxiliary maps

Note that the following Lie group isomorphisms occur in dimensions 3 and 4:

$$\begin{aligned} \mathrm{Spin}(3) &= SU(2), \\ \mathrm{Spin}^c(3) &= U(2), \\ \mathrm{Spin}(4) &= SU(2) \times SU(2), \\ \mathrm{Spin}^c(4) &= S(U(2) \times U(2)), \end{aligned}$$

where $S(U(2) \times U(2))$ means $SU(4) \cap (U(2) \times U(2))$. In the previous lecture we saw (although it was not stated exactly the same way) that the representation of Spin(3), respectively Spin^c(3), in S_3 is just the fundamental representation of $SU(2)$, respectively $U(2)$, in $\mathbb{C}^2$. Here, S_3 is the spin representation space for the Clifford algebra $Cl(\mathbb{R}^3)$ and the aforementioned representation of Spin group in S_3 is the one appearing in the definition of Spin structure on a manifold.

In what follows, it will be convenient to use the composition of the 'quantization map' into Clifford algebra and its spin representation, denoted by

$$\mathfrak{c} : \Lambda^2(\mathbb{R}^3) \to \mathrm{End}(S_3)$$

and generated by putting $\mathfrak{c}(e_k \wedge e_\ell) = \rho_3(e_k e_\ell)$ for any $k \neq \ell$. This map identifies $\Lambda^2 V$ with the space of traceless, skew-adjoint endomorphisms of S_3. Further, $\mathfrak{c}$ extends by complex linearity to a map from $\Lambda^2 V \otimes \mathbb{C}$ to the space of traceless endomorphisms of S_3, and the purely imaginary 2-forms then are mapped to self-adjoint endomorphisms.

Another important ingredient in Seiberg–Witten equations is the quadratic form

$$q : \varphi \in S_3 \to q(\varphi) = (\varphi \otimes \varphi^*) - \frac{1}{2}|\varphi|^2 \in \mathrm{End}(S_3)$$

defined by $q(\varphi)\psi = \langle \psi, \varphi\rangle\varphi - \frac{1}{2}|\varphi|^2\psi,\ \ \psi \in S_3$. Now, we have $S_3 \cong \mathbb{C}^2$ as Hermitian linear spaces. Accordingly, choosing an orthonormal basis in S_3, the quadratic form q can be expressed as

$$\varphi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto q(\varphi) = \begin{pmatrix} \frac{1}{2}(|\alpha|^2 - |\beta|^2) & \alpha\bar{\beta} \\ \bar{\alpha}\beta & \frac{1}{2}(|\beta|^2 - |\alpha|^2) \end{pmatrix}.$$

It is not difficult to see that q is Spin(3)-equivariant and, therefore, extends to a well-defined bundle map between the $\mathrm{Spin}^c(3)$ vector bundle $S^c(X)$ and the associated bundle $\mathrm{End}\, S^c(X)$.

Seiberg–Witten moduli space

Choose a Spin^c structure $s \in \mathcal{S}$ on X. Let A be a connection on the corresponding determinant line bundle $\det(S^c(X))$, F_A the curvature form of A, and $\psi \in \Gamma(S^c(X))$ a positive spinor field. Seiberg–Witten equations on X are a system of non-linear PDEs relating A and ψ,

$$\begin{aligned} &\mathfrak{c}(F_A) = q(\psi), \\ &D_A\psi = 0. \end{aligned} \tag{2.23}$$

It proves useful at times to be able to perturb the equations (2.23). To this end, consider a fixed closed differential 2-form μ on X. Then the perturbed Seiberg–Witten equations are then written as

$$\begin{aligned} &\mathfrak{c}(F_A + i\mu) = q(\psi), \\ &D_A\psi = 0. \end{aligned} \tag{2.23$'$}$$

Seiberg-Witten equations are equations from gauge theory, originating from advances in Mathematical Physics. Geometrically, they are defined using connections and sections on vector bundles, the Spin^c bundles. The natural symmetry, 'gauge symmetry', of these equations comes from the action of the group of automorphisms of the Spin^c bundle over X which cover the identity of the frame bundle of TX. Now the bundle automorphisms of $S^c(X)$ are sections of a bundle of groups whose fibre is the center S^1 of $\mathrm{Spin}^c(3) = U(2)$. This bundle of groups corresponds to the conjugation action of an abelian group S^1 on itself, $g \cdot h = ghg^{-1}$, and is, therefore, trivial. Thus the automorphisms of $S^c(X)$ are just the smooth maps from X to to S^1. Note that if u is an automorphism of $S^c(X)$ then the induced automorphism of the unitary line bundle $\lambda = \det S^c(X)$ is $u^2 \in C^\infty(X; S^1)$. The group $C^\infty(X; S^1)$ acts on the solutions of Seiberg–Witten equations by

$$(A, \psi) \mapsto (A + 2u\, du^{-1},\ u\,\psi), \quad u \in C^\infty(X; S^1), \tag{2.24}$$

defining an equivalence relation.

The Seiberg–Witten invariant is obtained by making a suitable count of solutions of (2.23). This, in particular, means that the pairs (A, ψ) related by 'gauge equivalence' (2.24) are not distinguished (as indeed one cannot otherwise expect a finite

non-trivial number of solutions). However, after dividing out by gauge symmetry, the space of solutions to Seiberg–Witten equations may still degenerate or depend on auxiliary choices. A basic principle for dealing with the problem is to analyse a 'generic' perturbation of the equations.

Remark. Here, the term 'generic' refers to the situation when a family of perturbations depends on a parameter. A Baire set of perturbations is one corresponding to an intersection of countably many open dense subsets in the range of the parameter. Then a **generic perturbation** is one whose parameter is contained in some fixed Baire set.

Denote by $\mathcal{M}$ the set of equivalence classes of the triples (s, A, ψ) where s is a Spin^c structure on X, and (A, ψ) is a solution of (2.23′) for the corresponding Spin^c bundle, and the equivalence relation is given by the formula (2.24). The set $\mathcal{M}$ is called the **moduli space** of solutions for Seiberg–Witten equations. In general, $\mathcal{M}$ depends on the choice of μ; however, we suppress this in the notation.

At this point, let us impose further conditions on the 3-manifold X. Assume that X is closed and that its first Betti number satisfies $b_1 > 1$ (the theory also works for the case $b_1 = 1$, with slightly less tidy results—we shall not discuss that here).

It is convenient to introduce a map Φ assigning to a triple (s, A, ψ) the 2-form $(2\pi i)^{-1}F_A$. By Chern–Weil theory, the De Rham cohomology class $[(2\pi i)^{-1}F_A] \in H^2_{DR}(X)$ is the image of Chern class $c_1(\lambda)$ in the natural map $H^2(X) \to H^2_{DR}(X)$. The latter map kills the torsion of $H^2(X)$, so we obtain a well-defined projection

$$\Phi : \mathcal{M} \to H^2(X)/\operatorname{Tor}. \tag{2.25}$$

Thus Φ gives a natural partition of the moduli space. Remember that there is a one-to-one correspondence between $\mathcal{S}$ and the image of Φ.

The following theorems are proved by the methods of gauge theory.

Theorem 2.26. *For a 3-manifold X as above, and for a generic perturbation μ of the Seiberg–Witten equations on X, the moduli space $\mathcal{M}$ has a natural structure of a real 0-dimensional, compact, oriented manifold.*

Theorem 2.27. *The oriented manifold $\mathcal{M}$ provided by the previous theorem and the projection $\Phi : \mathcal{M} \to H^2(X)$ depend only on the orientation preserving diffeomorphism class of X.*

A compact 0-dimensional manifold is simply a finite collection of points and the orientation is a choice of sign ± 1 for each of the points. According to Theorem 2.27, the equations (2.23′) generically have solutions only for finitely many Spin^c structures on X.

Definition. Let X be a 3-manifold as above. Seiberg–Witten invariant of X is a map $SW : \mathcal{S} \to \mathbb{Z}$ assigning to a Spin^c structure s on X the sum of signs of all the points in $\Phi^{-1}(s)$, $s \in \mathcal{S} \cong H^2(X)/\operatorname{Tor}$, where the signs are determined by the orientation on $\mathcal{M}$.

In summary, the discussed so far geometric constructions on an oriented Riemannian smooth 3-manifold X (satisfying certain restrictions) lead to the map SW depending only on the properties of X as a smooth manifold and expressible in terms of the topology of X. That is to say, the map SW is a *differential-topological invariant* of X.

ON THE WORK OF MENG AND TAUBES

Seiberg–Witten invariant and R-torsion

The Seiberg–Witten invariant map SW can be arranged as a single element of a certain cohomology group ring, via the following algebraic construction used by Meng and Taubes.

Fix a Spin^c structure $a \in \mathcal{S}$ on X. Recall that for any $s \in \mathcal{S}$ the difference of Chern classes for the corresponding determinant line bundles is even, $c_1(\lambda_s) - c_1(\lambda_a) = 2z$, $z \in H^2(X)/\operatorname{Tor}$. Define the function $SW_a : H^2(X)/\operatorname{Tor} \to \mathbb{Z}$, depending on the parameter a, by the formula

$$SW_a(z) = \sum_{s \in \mathcal{S},\, c_1(\lambda_s) - c_1(\lambda_a) = 2z} SW(s).$$

It is not difficult to check that (1) the above definition is equivariant with respect to the action of $H^2(X)/\operatorname{Tor}$, and (2) the function SW_a is finitely supported for every $a \in \mathcal{S}$. Here the action of $H^2(X)/\operatorname{Tor}$ is additive on itself and by the pull-back on the space of the integer valued functions on $H^2(X)/\operatorname{Tor}$.

In a more formal language, abbreviate $H^2(X)/\operatorname{Tor}$ as H and consider the ring $\mathbb{Z}[H]$ of finitely supported functions on H. Then $\mathbb{Z}[H]$ is equivalent to the integral group ring of H and the functions SW_a define an element in $\underline{SW} \in \mathbb{Z}[H]/H$.

We remarked at the end of the previous page that the expression of Seiberg–Witten invariant uses only the topology of X. On the other hand, the algebraic topological invariant $\tau(X)$, sometimes called Milnor–Turaev torsion, is an element in $\mathbb{Z}[H]/H$ and behaves similarly to $SW(X)$, e.g. under the connected sum operation. In fact, the following result is due to Meng and Taubes.

Theorem 2.28. *Let X be a closed oriented Riemannian manifold. Assume that $b_1(X) > 1$. Then $\underline{SW}(X) = \tau(X)$.*

The boundary (non-compactness) issues

It is possible to generalize the Seiberg–Witten invariant and Theorem 2.28 to some cylindrical end 3-manifolds.

Take X to be a connected oriented complete Riemannian manifold. But X is not compact: it may have finitely many ends diffeomorphic and isometric to half-cylinders $T^2 \times \mathbb{R}_+$, where T^2 denotes a torus. Then Euler characteristic of X vanishes. Again, we require that $b_1(X) > 1$.

To simplify the notation, let us write N for the disjoint union of the cross-sections of all the cylindrical ends of X. So, N is a disjoint (assumed non-empty) union of tori and X decomposes as a union

$$X = X_0 \cup_N (\mathbb{R}_+ \times N)$$

of two components identified along their common boundary N. We note in passing that the compact manifold X_0 has the same topology as X.

The main feature of the generalization of Seiberg–Witten theory to a cylindrical end manifold concerns imposing suitable "boundary conditions at infinity". Redefine $\mathcal{S}$ to

be the set of Spin^c structures on X for which the bundle $\det(S^c(X))$ is trivial over $\mathsf{pt}\times N$. Then a curvature form F_A on $\det(S^c(X))$ restricts to an exact 2-form over N and $\mathcal{S}$ is bijectively parametrized by the abelian group $\mathrm{Ker}(H^2(X) \to H^2(N))$.

Seiberg–Witten equations over X are complemented with the following "finite energy" type condition

$$\int_X |F_A|^2 d\mathrm{vol}_X < \infty. \tag{2.29}$$

Finally, the perturbation 2-form μ is constrained over the cross-section N to be non-zero and covariant constant, and $*\mu|_N = \nu|_N$ for some 1-form ν defined and closed on X.

Respectively, the Seiberg–Witten moduli space $\mathcal{M}$ over a cylindrical end manifold X is defined as a set of all the equivalence classes of solutions of (2.23) and (2.29).

With these modifications imposed, the assertions of Theorems 2.26 and 2.27 hold for $\mathcal{M}$, with one exception, regarding the compactness. In fact, the map Φ has to be recast as follows.

As was remarked, the curvature F_A of any connection A on a determinant line bundle coming from $\mathcal{S}$ restricts to an exact 2-form on the ends $\mathbb{R}_+ \times N$. It can be further shown by the analytic arguments that the condition (2.29) implies the exponential decay of the form F_A over the ends and that $(2\pi i)^{-1}F_A$ represents a well-defined map on the relative homology $H_2(X, N) \to \mathbb{Z}$ with integer coefficients. The latter map is identified with an element of the torsion-free part of integral cohomology group with compact support $H^2_c(X)/\operatorname{Tor}$. The map Φ as in (2.25) can now be defined via the natural inclusion $H^2_c(X)/\operatorname{Tor} \subset H^2(X)/\operatorname{Tor}$. Although the image of Φ is not identified with $\mathcal{S}$ in this case, there is a commutative diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{[(2\pi i)^{-1}F_A]} & H^2_c(X)/\operatorname{Tor} \\ {\scriptstyle c_1}\downarrow & & \downarrow \\ H^2(X) & \longrightarrow & H^2(X)/\operatorname{Tor}, \end{array}$$

where c_1 denotes the operation of taking the first Chern class of the determinant line bundle for $s \in \mathcal{S}$, of $[(s, A, \Phi)] \in \mathcal{M}$.

Now, even though the "global" moduli space $\mathcal{M}$ is not generally compact, it decomposes into compact pieces defined as the inverse images of points under Φ. And this is enough for the map SW to be well defined.

The invariant SW defines, with the help of the above commutative diagram, an element $\underline{SW} \in \mathbb{Z}(H)/H$. The following result is again due to Meng and Taubes (a choice of homology orientation for X is assumed).

Theorem 2.30. *Let X be a cylindrical end 3-manifold with $b_1(X) > 1$ and such that the cross-section of any cylindrical end is the torus. Then the Seiberg–Witten invariant $\underline{SW}$ equals the Milnor-Turaev torsion $\tau(X)$.*

NOTES. The descriptions of the 'quantization map' and quadratic form appearing in Seiberg–Witten equations are taken from [N]. The results on the Seiberg–Witten moduli space given here follow the

corresponding theorems in 4-dimensional theory [M], in conjunction with the remark in [MT] on the reduction to a 3-manifold by considering $S^1 \times X^3$. See also [MST] for a discussion of 3-dimensional Seiberg–Witten theory. The equivalence between 3-dimensional Seiberg–Witten invariant and Milnor–Turaev torsion is announced in [MT], see also [T] where an account of the torsion invariant can be found.

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Acknowledgement

I am grateful to Andrew Ranicki for the opportunity to speak at the reading seminar and for posting these notes on the WWW. It is a pleasure to acknowledge the support under William Gordon Seggie Brown Research Fellowship scheme. Nikolai Saveliev and me were exchanging discussions and materials on Seiberg–Witten theory over the past two years. — *A.K.* *12th November 1997.*