# NOTES ON REIDEMEISTER TORSION

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Reidemeister torsion (R-torsion for short) is an algebraic topology invariant which takes values in the multiplicative group of the units of a commutative ring. Although R-torsion has been largely subsumed in the more general theory of Whitehead torsion, it has retained a life of its own. There are applications to the structure theory of compact polyhedra and manifolds, especially in the odd dimensions (e.g. 3), knots and links, dynamical systems, analytic torsion, Seiberg-Witten theory, $\ldots$ .

The object of these notes is to provide a reasonably elementary introduction to R-torsion. The papers of Milnor [4], [5] are the classic references for R- and Whitehead torsion.

Here are some of the traditional algebraic topology invariants of a finite simplicial complex $X$ :

- the Betti numbers $b_*(X)$ (1871)
- the fundamental group $\pi_1(X)$ (1895)
- the homology groups $H_*(X)$, $b_i(X) = \dim H_i(X)$ (1925)

R-torsion is a more subtle invariant, being only defined for a finite simplicial complex $X$ and a commutative ring $A$ with a ring morphism $\mathbb{Z}[\pi_1(X)] \to A$ such that

$$H_*(X; A) \ = \ 0 \ ,$$

in which case the R-torsion is a unit of $A$

$$\Delta(X) \in A^\bullet \ .$$

R-torsion made its first appearance in 1935, in the work of Reidemeister [6] on the combinatorial classification of the 3-dimensional lens spaces by means of the based simplicial chain complex of the universal cover. In the same year Franz [3] (a student of Reidemeister) defined R-torsion in general, and used it to obtain the combinatorial classification of the high-dimensional lens spaces. R-torsion is a combinatorial invariant: finite simplicial complexes with isomorphic subdivisions have the same R-torsion. R-torsion is not a homotopy invariant: there exist homotopy equivalent spaces with different R-torsion, e.g. certain pairs of lens spaces. Whitehead published a false proof that R-torsion is a homotopy invariant in 1939, before going on to develop simple homotopy theory over the next 10 years: R-torsion is a simple homotopy invariant. In 1961 Milnor used R-torsion to disprove the Hauptvermutung, constructing two finite simplicial complexes without a common subdivision, but with homeomorphic polyhedra. It is now known that R-torsion is a topological invariant: finite simplicial complexes with homeomorphic polyhedra have the same R-torsion. The topological invariance of R-torsion was proved for

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manifolds by Kirby and Siebenmann in 1969, and for arbitrary simplicial complexes by Chapman in 1974. It followed that the 1930's combinatorial classifications of lens spaces were in fact also the topological classifications (= up to homeomorphism).

Roughly speaking, the R-torsion $\Delta(X)$ of a finite simplicial complex $X$ uses the action of the fundamental group $\pi_1(X)$ on the universal cover $\widetilde{X}$ to provide a global measure of the complexity of the simplicial structure of $X$. More precisely, the R-torsion is an invariant of the simplicial chain complex $C(\widetilde{X})$ of based f.g. free $\mathbb{Z}[\pi_1(X)]$-modules, the determinant of a square matrix obtained from the incidences of the simplices in $\widetilde{X}$. R-torsion makes essential use of the bases in the simplicial chain complex of the universal cover – the homology and homotopy groups do not see the geometric information encoded in the based chain complex. In fact, part of Reidemeister torsion is homotopy invariant, and this suffices for many applications (see Example 11 and Theorem 15 below). Whitehead torsion is more general than R-torsion, in that it also works for noncommutative rings (such as $\mathbb{Z}[\pi_1(X)]$) for which the determinant is not defined, but again it is an invariant of the based simplicial chain complex of the universal cover.

In practice, use $CW$ complexes and cell structures, along with the cellular chain complex.

Any one of the traditional invariants suffices to classify (oriented closed) 2-dimensional manifolds (= surfaces) up to homeomorphism:

*Every connected surface $M^2$ is homeomorphic to standard genus $g$ surface, with $b_1(M) = 2g$, and every homotopy equivalence $f : M^2 \to N^2$ of surfaces is homotopic to a homeomorphism.*

The situation is much more complicated for 3-manifolds! It is not even known if a 3-manifold $M^3$ with $\pi_*(M) = \pi_*(S^3)$ is homeomorphic to $S^3$ (Poincaré conjecture, 1904).

**Definition 1** (Tietze (1908)) The *lens spaces* are the closed oriented 3-dimensional manifolds

$$L(m,n) \;=\; S^3/\mathbb{Z}_m \;=\; \{(a,b) \in \mathbb{C}\times\mathbb{C} \,|\, |a|^2+|b|^2=1\}/(a,b) \sim (\zeta a, \zeta^n b)$$

with $\zeta = e^{2\pi i/m}$ a primitive $m^{\text{th}}$ root of unity, and $m$,$n$ coprime.

Thus $L(m,n)$ is the quotient of $S^3$ by the (fixed point) free action of the cyclic group $\mathbb{Z}_m$ of order $m$. The standard homotopy invariants of $L = L(m,n)$ are

$$\begin{aligned}
&\pi_1(L) \;=\; H_1(L) \;=\; \mathbb{Z}_m \;,\; \pi_i(L) \;=\; \pi_i(S^3) \text{ for } i \geq 2 \;,\\
&H_0(L) \;=\; H_3(L) \;=\; \mathbb{Z} \;,\; H_i(L) \;=\; 0 \text{ for } i \neq 0,1,3 \;,\\
&b_0(L) \;=\; b_3(L) \;=\; 1 \;,\; b_i(L) \;=\; 0 \text{ for } i \neq 0,3 \;.
\end{aligned}$$

**Example 2** $L(1,1) = S^3$, $L(2,1) = SO(3) = \mathbb{RP}^3$.

Reidemeister torsion is a generalization of the determinant of an invertible square matrix $\alpha = (a_{ij})$ with entries $a_{ij}$ in a commutative ring $A$

$$\det(\alpha) \in A^{\bullet} \;=\; \text{units of } A \;.$$

As usual, the *homology* of an $A$-module chain complex

$$C : \cdots \to C_{r+1} \xrightarrow{d} C_r \xrightarrow{d} C_{r-1} \to \cdots \to C_0 \;\; (d^2 = 0)$$

consists of the $A$-modules

$$H_r(C) \;=\; \ker(d : C_r \to C_{r-1})/\mathrm{im}(d : C_{r+1} \to C_r) \;(r \geq 0)\;.$$

The chain complex is *acyclic* if

$$H_r(C) \;=\; 0 \;(r \geq 0)\;.$$

It is a standard result of homological algebra that a chain complex $C$ with each $C_r$ free is acyclic if and only if there exists a *chain contraction*

$$\Gamma : 0 \simeq 1 : C \to C\;,$$

that is a collection of $A$-module morphisms $\Gamma : C_r \to C_{r+1}$ such that

$$d\Gamma + \Gamma d \;=\; 1 \;:\; C_r \to C_r \;(r \geq 0)\;.$$

**Lemma 3** If $C$ is an acyclic $A$-module chain complex with a chain contraction $\Gamma : 0 \simeq 1 : C \to C$ then the $A$-module morphism

$$f \;=\; d + \Gamma \;=\; \begin{pmatrix} d & 0 & 0 & \dots \\ \Gamma & d & 0 & \dots \\ 0 & \Gamma & d & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \;:\; C_{odd} \to C_{even}$$

is an isomorphism, with

$$C_{odd} \;=\; C_1 \oplus C_3 \oplus C_5 \oplus \dots\;,\; C_{even} \;=\; C_0 \oplus C_2 \oplus C_4 \oplus \dots\;.$$

**Proof** The $A$-module morphism

$$g \;=\; d + \Gamma \;=\; \begin{pmatrix} \Gamma & d & 0 & \dots \\ 0 & \Gamma & d & \dots \\ 0 & 0 & \Gamma & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \;:\; C_{even} \to C_{odd}$$

is such that both the composites

$$fg : C_{even} \to C_{even}\;,\; gf : C_{odd} \to C_{odd}$$

are of the type

$$\begin{pmatrix} 1 & 0 & 0 & \dots \\ \Gamma^2 & 1 & 0 & \dots \\ 0 & \Gamma^2 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and hence isomorphisms. It follows that $f$ is an isomorphism, with inverse

$$f^{-1} \;=\; (gf)^{-1}g \;=\; g(fg)^{-1} \;:\; C_{even} \to C_{odd}\;.$$

From now on, only finite chain complexes $C$ are considered, with

$$C : \dots \to 0 \to C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} C_{n-2} \to \dots \to C_0$$

for some $n \geq 0$.

**Definition 4** The *Reidemeister torsion* of an acyclic chain complex $C$ of based f.g. free $A$-modules is

$$\Delta(C) \;=\; \det(d + \Gamma : C_{odd} \to C_{even}) \in A^{\bullet}$$

for any chain contraction $\Gamma : 0 \simeq 1 : C \to C$.

**Example 5** For 1-dimensional $C$

$$\Delta(C) \ = \ \det(d : C_1 \to C_0) \in A^\bullet \ .$$

How does one associate an acyclic chain complex to a space $X$? In the first instance, recall the standard way of associating a chain complex to a connected $CW$ complex

$$X \ = \ \bigcup_{r=0}^{\infty} \bigcup e^r \ .$$

Lift each cell $e^r \subset X$ to a cell $\widetilde{e}^r \subset \widetilde{X}$ in the universal cover of $X$, which thus has a $\pi_1(X)$-equivariant cell structure

$$\widetilde{X} \ = \ \bigcup_{g \in \pi_1(X)} \bigcup_{r=0}^{\infty} \bigcup g\widetilde{e}^r$$

with $\pi_1(X)$ acting on $\widetilde{X}$ as the group of covering translations

$$\pi_1(X) \times \widetilde{X} \to \widetilde{X} \ ; \ (g, x) \mapsto gx \ .$$

The group ring $\mathbb{Z}[\pi_1(X)]$ consists of the finite linear combinations

$$\sum_{g \in \pi_1(X)} n_g g \ (n_g \in \mathbb{Z}) \ .$$

The cellular chain complex of $\widetilde{X}$

$$C(\widetilde{X}) : \cdots \to C(\widetilde{X})_{r+1} \xrightarrow{d} C(\widetilde{X})_r \xrightarrow{d} C(\widetilde{X})_{r-1} \to \cdots \to C(\widetilde{X})_0$$

is a chain complex of based free $\mathbb{Z}[\pi_1(X)]$-modules, with

$$\begin{aligned} C(\widetilde{X})_r \ &= \ H_r(\widetilde{X}^{(r)}, \widetilde{X}^{(r-1)}) \\ &= \ \text{based f.g. free } \mathbb{Z}[\pi_1(X)]\text{-module generated by the } r\text{-cells } e^r \subset X \end{aligned}$$

with $\widetilde{X}^{(r)}$ the induced cover of the $r$-skeleton of $X$

$$X^{(r)} \ = \ \bigcup_{j \leq r} \bigcup e^j \subseteq X \ .$$

The basis elements are only determined by the cell structure of $X$ up to multiplication by $\pm g$ ($g \in \pi_1(X)$). The geometric action of $\pi_1(X)$ on $\widetilde{X}$ induces the algebraic action of $\mathbb{Z}[\pi_1(X)]$ on $C(\widetilde{X})$. Now

$$H_0(C(\widetilde{X})) \ = \ H_0(\widetilde{X}) \ = \ \mathbb{Z}$$

so $C(\widetilde{X})$ is not acyclic, and does not have a Reidemeister torsion. The trick is to find a commutative ring $A$ with a ring morphism $f : \mathbb{Z}[\pi_1(X)] \to A$ such that the induced $A$-module chain complex

$$C(X; A) \ = \ A \otimes_{\mathbb{Z}[\pi_1(X)]} C(\widetilde{X})$$

is acyclic.

**Definition 6** The *Reidemeister torsion* of a finite $CW$ complex $X$ with respect to a ring morphism $f : \mathbb{Z}[\pi_1(X)] \to A$ to a commutative ring $A$ such that $H_*(X; A) = 0$ is

$$\Delta(X) \ = \ \Delta(C(X; A)) \in A^\bullet / \{f(\pm\pi_1(X))\} \ .$$

**Example 7** Let $X = L(m,n)$, and choose a generator $t \in \pi_1(X) = \mathbb{Z}_m$, so that the group ring can be written as

$$\mathbb{Z}[\mathbb{Z}_m] \;=\; \mathbb{Z}[t,t^{-1}]/(1-t^m)\ .$$

Since $m,n$ are coprime there exist integers $a,b$ such that

$$an + bm \;=\; 1\ .$$

Now $L(m,n)$ has a $CW$ decomposition with 1 cell $e^j$ in each dimension $j \leq 3$

$$L(m,n) \;=\; e^0 \cup e^1 \cup e^2 \cup e^3$$

which lifts to a $\mathbb{Z}_m$-equivariant $CW$ structure on the universal cover

$$\widetilde{L(m,n)} \;=\; S^3 \;=\; \bigcup_{k=0}^{m-1} (t^k\widetilde{e}^0 \cup t^k\widetilde{e}^1 \cup t^k\widetilde{e}^2 \cup t^k\widetilde{e}^3)$$

with $m$ cells $t^k\widetilde{e}^j$ in each dimension $j \leq 3$. The cellular chain complex of based f.g. free $\mathbb{Z}[\mathbb{Z}_m]$-modules

$$C = C(\widetilde{L(m,n)}) : \dots \to 0 \to \mathbb{Z}[\mathbb{Z}_m] \overset{1-t^a}{\to} \mathbb{Z}[\mathbb{Z}_m] \overset{N}{\to} \mathbb{Z}[\mathbb{Z}_m] \overset{1-t}{\to} \mathbb{Z}[\mathbb{Z}_m]$$

with $N = 1+t+\dots+t^{m-1} \in \mathbb{Z}[\mathbb{Z}_m]$. Here are two ways of associating Reidemeister torsion to $L(m,n)$.

(i) For each primitive $m^{\text{th}}$ root $\zeta$ of 1 in $\mathbb{C}$ there is defined a ring morphism

$$f_\zeta \;:\; \mathbb{Z}[\mathbb{Z}_m] \to \mathbb{C}\ ;\ t \mapsto \zeta$$

such that $H_*(L(m,n);\mathbb{C}) = 0$ (since $1+\zeta+\zeta^2+\dots+\zeta^{m-1} = 0$), with the Reidemeister torsion given by

$$\Delta_\zeta(L(m,n)) \;=\; \Delta(C(L(m,n);\mathbb{C})) \;=\; (1-\zeta)(1-\zeta^a) \in \mathbb{C}^\bullet/\{f(\pm\mathbb{Z}_m)\}\ .$$

Since

$$\Delta_{\overline{\zeta}}(L(m,n)) \;=\; \overline{\Delta_\zeta(L(m,n))}$$

only the complex conjugacy class of $\zeta$ is relevant, giving $\phi(m)/2$ possibilities for $f$, with

$$\phi(m) \;=\; \text{Euler's function} = |\{n < m \text{ coprime to } m\}|\ .$$

(ii) The ring morphism

$$f : \mathbb{Z}[\mathbb{Z}_m] \to A = \mathbb{Q}[\mathbb{Z}_m]/(N)$$

induces from $C$ the based f.g. free $A$-module chain complex

$$A \otimes_{\mathbb{Z}[\mathbb{Z}_m]} C \;:\; \dots \to 0 \to A \overset{1-t^a}{\to} A \overset{0}{\to} A \overset{1-t}{\to} A\ .$$

(Direct proof that $1-t \in A^\bullet$ :

$$(1-t)(1+2t+3t^2+\dots+mt^{m-1}) \;=\; N-m \in \mathbb{Z}[\mathbb{Z}_m]$$

and similarly for $1-t^a$). Thus $H_*(L(m,n);A) = 0$ and the Reidemeister torsion is given by

$$\Delta(L(m,n)) \;=\; (1-t)(1-t^a) \in A^\bullet/\{f(\pm\mathbb{Z}_m)\}\ .$$

Note that $\Delta(L(m,n))$ determines $\Delta_\zeta(L(m,n))$, with

$$f_\zeta \;:\; \mathbb{Z}[\mathbb{Z}_m] \overset{f}{\to} A \overset{t\to\zeta}{\to} \mathbb{C}\ ,$$

and

$$\Delta_\zeta(L(m,n)) \;=\; f_\zeta(\Delta(L(m,n))) \in \mathbb{C}^\bullet/\{f(\pm\mathbb{Z}_m)\}\ .$$

In fact, the function

$$A^\bullet \ \to \ \bigoplus_{\phi(m)/2} \mathbb{C}^\bullet \ ; \ u \mapsto \bigoplus f_\zeta(u)$$

is an injection, and $\Delta(L(m,n))$ is determined by the $\phi(m)/2$ complex numbers $\Delta_\zeta(M)$ ($\zeta = e^{2\pi i\ell/m}$, $(\ell,m)=1$, $\mathrm{Im}(\zeta)\geq 0$).

The Reidemeister torsion of $L(m,n)$ fishes out the topologically relevant part of $n$ from $L(m,n)$ – note that $H_*(C) = H_*(S^3)$ is the same for all the $L(m,n)$'s, so homology alone is quite inadequate.

If $h : L(m',n') \to L(m,n)$ is a homotopy equivalence of lens spaces then

$$m \ = \ m'$$

and

$$h_* : \pi_1(L(m,n)) = \mathbb{Z}_m \to \pi_1(L(m,n')) = \mathbb{Z}_m \ ; \ t \to t^r$$

for some $r \in \mathbb{Z}_m^\bullet$.

**Theorem 8** (Franz, Rueff, Whitehead (1940))
(i) *The following conditions are equivalent:*

- $L(m,n)$ *is homotopy equivalent to* $L(m,n')$,
- $n \equiv \pm n' r^2 (\bmod\, m)$ *for some* $r \in \mathbb{Z}_m^\bullet$.

(ii) *The following conditions are equivalent:*

- $L(m,n)$ *is homeomorphic to* $L(m,n')$,
- $n \equiv \pm n' r^2 (\bmod\, m)$ *with* $r \equiv 1$ *or* $n$ $(\bmod\, m)$,
- *there exists a homotopy equivalence* $h : L(m,n') \to L(m,n)$ *such that*
$$h_*\Delta_\zeta(L(m,n')) \ = \ \Delta_\zeta(L(m,n)) \in \mathbb{C}^\bullet/\{\pm\mathbb{Z}_m\}$$
*for each primitive* $m^{th}$ *root* $\zeta$ *of 1,*
- *there exists a homotopy equivalence* $h : L(m,n') \to L(m,n)$ *such that*
$$h_*\Delta(L(m,n')) \ = \ \Delta(L(m,n)) \in \mathbb{Q}[\mathbb{Z}_m]^\bullet/\{\pm\mathbb{Z}_m\} \ .$$

**Proof** See deRham, Kervaire, Maumary [2] and Cohen [1] for detailed accounts. The basic idea is that there exists a homotopy equivalence

$$h \ : \ M' \ = \ L(m,n') \to M \ = \ L(m,n)$$

such that $h_*(t) = t^r$ if and only if there exists a unit $u \in \mathbb{Z}[\mathbb{Z}_m]^\bullet$ such that

$$(1-t^a)u \ = \ N_r(1-t^{a'r}) \in \mathbb{Z}[\mathbb{Z}_m]$$

with $N_r = 1+t+\dots+t^{r-1}$, $a'n'+b'm=1$, in which case the induced $\mathbb{Z}[\mathbb{Z}_m]$-module chain equivalence is given by

$$\begin{array}{ccc} h_*C(\widetilde{M}') & : & \Lambda \xrightarrow{1-t^{a'r}} \Lambda \xrightarrow{N} \Lambda \xrightarrow{1-t^r} \Lambda \\ \Big\downarrow \widetilde{h} & & \Big\downarrow u \qquad \Big\downarrow N_r \qquad \Big\downarrow N_r \qquad \Big\downarrow 1 \\ C(\widetilde{M}) & : & \Lambda \xrightarrow{1-t^a} \Lambda \xrightarrow{N} \Lambda \xrightarrow{1-t} \Lambda \end{array}$$

The algebraic condition in (i) is precisely the necessary and sufficient condition for the existence of such $r,u$. The homotopy equivalence $h$ is homotopic to a homeomorphism if and only if $u$ is of the form $\pm t^j$, if and only if any one of the algebraic conditions in (ii) is satisfied.

**Example 9** $L(5,1)$ is not homotopy equivalent to $L(5,2)$.

**Example 10** $L(7,1)$ is homotopy equivalent but not homeomorphic to $L(7,2)$.

The Reidemeister torsion $\Delta(C) \in A^\bullet$ is very sensitive to the choice of basis for the acyclic f.g. free $A$-module chain complex $C$ : if $C'$ is the same chain complex with different basis then

$$\Delta(C')\Delta(C)^{-1} \;=\; \det(\alpha_0)\det(\alpha_1)^{-1}\det(\alpha_2)\cdots \in A^\bullet$$

with $\alpha_r$ the determinant of the change of basis matrix from $C_r$ to $C'_r$.

In the applications of the Reidemeister torsion $\Delta(C) \in A^\bullet$ to topology the acyclic based f.g. free $A$-module chain complex $C$ is of the form

$$C \;=\; A \otimes_B D$$

with $D$ a based f.g. free $B$-module chain complex, and $B \to A$ a ring morphism. If $B$ is commutative the residue class

$$[\Delta(C)] \in \operatorname{coker}(B^\bullet \to A^\bullet)$$

is in fact a chain homotopy invariant of $D$ (by the above change of basis formula, with each $\det(\alpha_r) \in \operatorname{im}(B^\bullet \to A^\bullet)$).

**Example 11** The residue class of the Reidemeister torsion of a finite $CW$ complex $X$ with $\pi_1(X)$ abelian and $H_*(X;A) = 0$

$$[\Delta(X)] \in \operatorname{coker}(\mathbb{Z}[\pi_1(X)]^\bullet \to A^\bullet)$$

is a homotopy invariant. (This generalizes to the case of nonabelian $\pi_1(X)$ using Whitehead torsion, and replacing the group of units $\mathbb{Z}[\pi_1(X)]^\bullet$ by the Whitehead group $Wh(\pi_1(X))$.)

In many cases, it is possible to express the residue class $[\Delta(C)]$ directly in terms of the homology $H_*(D)$, as was first worked out by Milnor [4].

**Definition 12** An $A$-module $E$ is *stably f.g. free* if there exists an isomorphism

$$E \oplus A^m \cong A^n$$

for some $m, n \geq 0$.

**Lemma 13** If $C$ is an acyclic f.g. free $A$-module chain complex then each

$$B_r \;=\; \operatorname{im}(d : C_{r+1} \to C_r)$$

is a stably f.g. free $A$-module with a stably f.g. free direct complement $E_r \subset C_r$ such that $d| : E_r \to B_{r-1}$ is an isomorphism.

**Proof** Let $\Gamma : 0 \simeq 1 : C \to C$ be a chain contraction of $C$ with $\Gamma : C_r \to C_{r+1}$ such that

$$d\Gamma + \Gamma d \;=\; 1 \;:\; C_r \to C_r \;(r \geq 0)\;.$$

Use $\Gamma$ to define a chain contraction $\Gamma' : 0 \simeq 1 : C' \to C'$ of the chain complex

$$C' \;:\; \cdots \to 0 \to B_r \to C_r \to C_{r-1} \to \cdots \to C_1 \to C_0$$

with $\Gamma' = \Gamma$ on $C'_j = C_j$ $(0 \leq j \leq r-1)$ and

$$\Gamma' \;:\; C'_r \;=\; C_r \to C'_{r+1} \;=\; B_r \;;\; x \mapsto d\Gamma(x)\;.$$

It now follows from the isomorphism

$$d' + \Gamma' \;:\; B_r \oplus C_{r-1} \oplus C_{r-3} \oplus \ldots \;\cong\; C_r \oplus C_{r-2} \oplus C_{r-4} \oplus \ldots$$

that $B_r$ is stably f.g. free. The $A$-module defined by

$$E_r \;=\; \mathrm{im}(\Gamma : C_{r-1} \to C_r) \subseteq C_r$$

is such that $C_r = B_r \oplus E_r$ with $d| : E_r \to B_{r-1}$ an isomorphism.

For simplicity assume now that the rings $A$ is such that stably f.g. free $A$-modules are actually f.g. free, such as a field or PID (= principal ideal domain).

**Definition 14** An *internal* basis of an acyclic f.g. free $A$-module chain complex $C$ is one which is obtained by extending bases for the f.g. free $A$-modules $B_r = \mathrm{im}(d : C_r \to C_{r-1})$ to bases for $C_r = B_r \oplus E_r$ by means of the isomorphisms $d| : E_r \to B_{r-1}$.

**Theorem 15** (Milnor [4]) *The Reidemeister torsion of an acyclic based f.g. free $A$-module chain complex $C$ is*

$$\Delta(C) \;=\; \det(\alpha_0)\det(\alpha_1)^{-1}\det(\alpha_2)\cdots \in A^\bullet$$

*with $\alpha_i$ the matrix expressing the change of basis on $C_i$ from an internal basis to the given basis. Independent of the choice of internal basis.*

**Proof** Consider first the elementary case when $C$ is of the type

$$E \;:\; \cdots \to 0 \to E_{r+1} \xrightarrow{d} E_r \to 0 \to \dots \;,$$

with $d : E_{r+1} \to E_r$ an isomorphism. If $e_{r+1}, e_r$ are the given bases of $E_{r+1}, E_r$ then $e_{r+1}, d(e_{r+1})$ are internal bases, and

$$\det(\alpha_r) \;=\; \det(d) \;,\; \det(\alpha_{r+1}) \;=\; 1 \in A^\bullet \;.$$

The general case reduces to the special case, since $C$ is isomorphic as an unbased chain complex to a direct sum of elementary complexes.

Note analogy with the Euler characteristic $\chi(M) = \sum\limits_{i=0}^{\infty}(-)^i b_i(M)$.

**Example 16** (Milnor [5]) For any field $F$ let $F[t, t^{-1}]$ be the Laurent polynomial ring (which is a PID), with quotient field

$$F(t) \;=\; \{\frac{p(t)}{q(t)} \,|\, p(t), q(t) \in F[t, t^{-1}], q(t) \neq 0\} \;.$$

A f.g. free $F[t, t^{-1}]$-module chain complex $B$ induces a f.g. free $F(t)$-module (= vector space) chain complex

$$C \;=\; F(t) \otimes_{F[t,t^{-1}]} B$$

such that $H_*(C) = 0$ if and only if

$$\dim_F H_*(B) < \infty \;.$$

Assume $H_*(C) = 0$ and give $C$ a basis by inducing from an arbitrary basis for each $B_r$ $(r \geq 0)$, so that $\Delta(C) \in F(t)^\bullet$ is defined. The action of $t$ on $H_*(B)$ defines automorphisms of finite dimensional $F$-vector spaces

$$h \;:\; H_r(B) \to H_r(B) \;;\; x \mapsto tx \;.$$

By the Cayley-Hamilton theorem $C$ is chain equivalent to the f.g. free $F[t, t^{-1}]$-module chain complex $D$ with

$$\begin{aligned} d_D \;=\; \begin{pmatrix} 0 & h - t \\ 0 & 0 \end{pmatrix} \;:\; D_r \;&=\; H_r(B)[t, t^{-1}] \oplus H_{r-1}(B)[t, t^{-1}] \\ \to\; D_{r-1} \;&=\; H_{r-1}(B)[t, t^{-1}] \oplus H_{r-2}(B)[t, t^{-1}] \;. \end{aligned}$$

If $D$ is given a basis by inducing from an arbitrary basis for each $H_r(B)$ $(r \geq 0)$ then the Reidemeister torsion of $D$ is given by

$$\Delta(D) \;=\; p_0(t)p_1(t)^{-1}p_2(t)\cdots \in F(t)^\bullet \;=\; F(t)\backslash\{0\}$$

with

$$p_r(t) \;=\; \det(h - t : H_r(B)[t,t^{-1}] \to H_r(B)[t,t^{-1}]) \in F[t,t^{-1}]$$

the characteristic polynomial of $h : H_r(B) \to H_r(B)$. Since $C, D$ are related by an $F[t,t^{-1}]$-module chain equivalence $C \simeq D$

$$\Delta(C)\Delta(D)^{-1} \in F[t,t^{-1}]^\bullet \subset F(t)^\bullet$$

with

$$F[t,t^{-1}]^\bullet \;=\; \{at^j \,|\, a \in F^\bullet, j \in \mathbb{Z}\} \ (F^\bullet = F\backslash\{0\})\ .$$

The residue class

$$\Delta(C) \;=\; \Delta(D) \;=\; p_0(t)p_1(t)^{-1}p_2(t)\cdots \in F(t)^\bullet/F[t,t^{-1}]^\bullet$$

is independent of the choice of basis of $B$, and is a homology invariant of $B$.

In the original application of Reidemeister torsion to 3-manifolds $M$ the fundamental group $\pi_1(M)$ was finite. In the applications to knots and links the fundamental group is infinite.

**Example 17** (Milnor [4],[5]) Given a knot $k : S^1 \subset S^3$ let

$$X \;=\; \mathrm{closure}(S^3\backslash k(S^1) \times D^2)$$

be the knot exterior, which has a cell structure of the type

$$X \;=\; e^0 \cup \bigcup_{n+1} e^1 \cup \bigcup_n e^2\ .$$

There is a homology equivalence $X \to S^1$ inducing a surjection $\pi_1(X) \to \mathbb{Z}$, which classifies the canonical infinite cyclic cover $\overline{X}$ of $X$, with generating covering translation $h : \overline{X} \to \overline{X}$. The cellular $\mathbb{Z}[t,t^{-1}]$-module chain complex of $\overline{X}$ is of the form

$$C(\overline{X}) \;:\; \cdots \to 0 \to \mathbb{Z}[t,t^{-1}]^n \xrightarrow{d_2} \mathbb{Z}[t,t^{-1}]^{n+1} \xrightarrow{d_1} \mathbb{Z}[t,t^{-1}]\ ,$$

with $h_* = t : C(\overline{X}) \to C(\overline{X})$. The Alexander polynomial of $k$

$$A(t) \;=\; a_0 + a_1t + \cdots + a_mt^m \in \mathbb{Z}[t,t^{-1}] \ (a_0, a_m \neq 0 \in \mathbb{Z})$$

is the highest common factor of the $n \times n$ subdeterminants of the $(n+1) \times n$ matrix of $d_2$. For any field $F$

$$H_0(\overline{X};F) \;=\; F\ ,\ \dim_F H_1(\overline{X};F) \;=\; m\ ,\ H_j(\overline{X};F) \;=\; 0\ (j \geq 2)\ .$$

The characteristic polynomial of the automorphism $h_* : H_1(\overline{X};F) \to H_1(\overline{X};F)$

$$p(t) \;=\; \det(h_* - t : H_1(\overline{X};F)[t,t^{-1}] \to H_1(\overline{X};F)[t,t^{-1}]) \in F[t,t^{-1}]$$

is related to the Alexander polynomial $A(t)$ by

$$p(t) \;=\; \frac{A(t)}{a_m} \in F[t,t^{-1}]\ .$$

(It is well-known that $m$ must be even, and that $A(t) = t^mA(t^{-1})$, with $a_0 = a_m$, $a_1 = a_{m-1}, \ldots$). The composite

$$f \;:\; \mathbb{Z}[\pi_1(X)] \to \mathbb{Z}[\mathbb{Z}] \;=\; \mathbb{Z}[t,t^{-1}] \to F[\mathbb{Z}] \;=\; F[t,t^{-1}] \to F(t)$$

is such that
$$H_*(X;F(t)) \;=\; F(t)\otimes_{F[t,t^{-1}]} H_*(\overline{X};F) \;=\; 0\ ,$$
since
$$p(t)H_1(\overline{X};F) \;=\; 0\ ,\ (1-t)H_0(\overline{X};F)=0\ .$$
The $F[t,t^{-1}]$-module chain complex $F[t,t^{-1}]\otimes_{\mathbb{Z}[t,t^{-1}]} C(\overline{X})$ is chain equivalent to
$$C\ :\ \cdots \to 0 \to F[t,t^{-1}]^m \xrightarrow{(h_*-t)\oplus 0} F[t,t^{-1}]^m \oplus F[t,t^{-1}] \xrightarrow{0\oplus(1-t)} F[t,t^{-1}]\ .$$
The residue class of the Reidemeister torsion of $X$
$$\Delta(X) \;=\; \Delta(C) \;=\; \frac{1-t}{p(t)} \in F(t)^\bullet/F[t,t^{-1}]^\bullet$$
is a homotopy invariant (cf. Example 16) which is determined by the Alexander polynomial $A(t)$. Conversely, if $\mathrm{char}(F)\neq 0$ the Reidemeister torsion determines the proportions $a_0 : a_1 : \cdots : a_m$ of the coefficients in $A(t)$.

See Turaev [7] for a survey of the applications of Reidemeister torsion to knots and links.

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