Nonlinear dynamics over rough topography: homogeneous and stratified quasi-geostrophic theory

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The weakly nonlinear dynamics of quasi-geostrophic flows over a one-dimensional, periodic or random, small-scale topography is investigated using an asymptotic approach. Averaged (or homogenised) evolution equations which account for the flow-topography interaction are derived for both homogeneous and continuously stratified quasi-geostrophic fluids. The scaling assumptions are detailed in each case; for stratified fluids, they imply that the direct influence of the topography is confined within a thin bottom boundary layer, so that it is through a new bottom boundary condition that the topography affects the large-scale flow. For both homogeneous and stratified fluids, a single scalar function entirely encapsulates the properties of the topography that are relevant to the large-scale flow: it is the correlation function of the topographic height in the homogeneous case, and a linear transform thereof in the continuously stratified case.

Some properties of the averaged equations, including their Hamiltonian structure, are discussed. Explicit nonlinear solutions in the form of one-dimensional travelling waves can be found. In the homogeneous case, previously studied by Volosov, they obey a second-order differential equation; in the stratified case on which we focus they obey a nonlinear pseudo-differential equation, which reduces to the Peierls–Nabarro equation for sinusoidal topography. The known solutions to this equation provide examples of nonlinear periodic and solitary waves in continuously stratified fluid over topography.

The influence of bottom topography on large-scale baroclinic instability is also examined using the averaged equations: they allow a straightforward extension of Eady's model which demonstrates the stabilising effect of topography on baroclinic instability.

1. Introduction

The effect of bottom topography on large-scale ocean dynamics has been studied using a variety of modelling hypotheses. Starting with Rhines & Bretherton (1973), a number of authors have considered topographies whose scale is much smaller than the typical scale of motion. This allows asymptotic techniques — essentially homogenisation techniques — to be employed to derive effective equations of motion in which only the averaged, large-scale effect of the topography is represented. Recent papers by Reznik & Tsybaneva (1999) and Bobrovich & Reznik (1999) present detailed analyses of this kind for, respectively, two-layer and stratified quasi-geostrophic flows over one-dimensional topography; the reader is referred to these papers for further background on the problem and for references.

Most of the previous results concern linear waves propagating over one-dimensional topography. A series of papers by Volosov and Zdhanov is however devoted to a nonlinear theory: extending the asymptotic approach of Rhines & Bretherton (1973), they derive evolution equations for weakly nonlinear motion over small-scale topography in the quasi-geostrophic regime, using homogeneous (Volosov 1976*c*,*a*,*b*), two-layer (Volosov & Zhdanov 1982) and continuously stratified models (Volosov & Zhdanov 1983). From these equations it is then easy to obtain closed-form solutions representing one-dimensional periodic travelling waves, which directly generalise linear Rossby and topographic waves, and solitary waves.

The present paper is similar in spirit to those of Volosov and Zdhanov. Its main novel result is an asymptotic theory for the weakly nonlinear motion of a continuously stratified quasi-geostrophic fluid over a small-scale, one-dimensional topography. This can be viewed as an extension to the nonlinear regime of some of the linear results of Bobrovich & Reznik (1999). As is detailed in that paper, the dynamical regime is mainly specified by the relationship between three key length scales, namely the scale of motion L, the scale of the topography L_{t} and the internal Rossby radius of deformation L_{i}, defined by $L_{\mathrm{i}} = NH/f$, where N, H and f are typical values of the Brunt-Väisälä frequency, ocean depth and Coriolis parameter, respectively. A crucial assumption, common to all the papers cited above, is that of a scale separation between motion and topography, explicitly expressed as

$$\epsilon = \frac{L_{\mathrm{t}}}{L} \ll 1.$$

For a stratified ocean, the relation between L_{t} and L_{i} is also important. Here, we assume that $L_{\mathrm{i}} \sim L$, so that $L_{\mathrm{t}} \ll L_{\mathrm{i}}$. This implies that the separation between the large scale of the leading-order motion and the small scale of the topography (and hence of the small-amplitude topography-induced motion) which is assumed in the horizontal holds also in the vertical: the topography-induced motion has vertical scale fL_{t}/N, much smaller than the total ocean depth H and is in fact localised within a boundary layer.

The scaling assumption $L_{\mathrm{t}} \ll L_{\mathrm{i}}$ makes our treatment of the stratified quasi-geostrophic dynamics markedly distinct from that of Volosov & Zhdanov (1983). In that paper, the different assumption $L_{\mathrm{i}} \lesssim L_{\mathrm{t}}$ is made. The consistency of the asymptotic analysis then requires the leading-order motion to be independent of the vertical coordinate, so that the averaged equations are barotropic. Here, in contrast, arbitrary vertical structures are allowed, and the averaged equations are fully three dimensional.

Once the relative magnitude of the three length scales is fixed, the other parameters in the model may be chosen to obtain a distinguished limit in which all physical effects (topography, β-effect, nonlinearity) have a similar importance while a consistent asymptotic solution can be developed. In the continuously stratified case, this turns out to require a ratio h/H of the topography height to the total ocean depth that scales like $\epsilon^{1/2}$ and a (suitably non-dimensionalised) velocity amplitude that scales like ϵ. With this scaling, closed averaged equations can be derived: they are given by the usual (linearised) potential-vorticity conservation in the fluid interior, with a bottom boundary condition provided by a system of two coupled nonlinear equations for the bottom potential temperature and the fluid-parcel displacement across isobaths. This system presents interesting similarities and differences with the corresponding system of averaged equations found in the homogeneous case (which assumes that $h/H = O(1)$ rather than $O(\epsilon^{1/2})$). For instance, the characteristics of the topography relevant to the large-scale motion are encapsulated in a single function; this is the correlation function of the topography height in the homogeneous case, but a linear transform (related to the Hilbert transform) of the correlation function in the stratified case.

To allow comparison between the effect of topography in homogeneous and stratified fluids, the paper starts with a derivation of the averaged equations in the homogeneous case (§2.1). The results of Volosov (1976*a,b,c*) are thus recovered, with minor extensions such as the inclusion of a finite radius of deformation and a discussion of the Hamiltonian structure of the averaged equations (§2.2).

The central result of the paper, namely the averaged equations for stratified quasi-geostrophic flow over small-scale topography is derived in §3.1. After a brief presentation of a Hamiltonian structure for these equations (§3.2), we consider a reduced model of particular interest (§3.3). This model arises when, in the absence of β-effect, the vorticity in the fluid interior vanishes identically. The dynamics is then controlled entirely by the evolution of the potential temperature and particle displacement on the bottom boundary. This model with trivial interior dynamics, which may be viewed as resulting from the inclusion of small-scale topography in the so-called surface quasi-geostrophic model (Held, Pierrehumbert, Garner & Swanson 1995), is governed by evolution equations involving a pseudo-differential operator related to the Hilbert transform.

The waves propagating in the model, supported by both the topography and the β-effect are discussed. In the linear approximation, a dispersion relation obtained by Bobrovich & Reznik (1999) is recovered (§4.1). When the nonlinearity is taken into account, the situation is somewhat involved, since the travelling-wave problem (one-dimensional in particular) is governed by a nonlinear pseudo-differential equation. However, we find closed-form solutions for a sinusoidal topography under the assumption that $\beta = 0$ (i.e. for the surface quasi-geostrophic model; §4.2). In this case, the nonlinear pseudo-differential equation to solve reduces to the Peierls–Nabarro equation whose periodic solutions have been studied by Toland (1997). Of particular physical interest is the solitary-wave solution (or, for the particle displacements, kink solution) that can be found as a limit of periodic solutions.

An issue that can be examined using the averaged model derived in this paper is the influence of small-scale topography on large-scale-flow instability, in particular on baroclinic instability. This can be done straightforwardly by including in the model a large-scale, vertically sheared flow and by carrying out a spectral stability analysis. In §5, we apply this approach to examine how the simplest model of baroclinic instability of a continuously stratified fluid, namely Eady's model, is affected by topography. The results complement those recently obtained by Benilov (2001) who addressed the same issue using Phillips' two-layer model.

A remark should be made about the limitations of the approach used in this paper. As is usual when formal asymptotics is used, the averaged model is accurate in describing the behaviour of the full system over finite, although large, spatial and temporal scales. This remark is particularly relevant when one considers random topographies, with a height field given by a non-degenerate random function (e.g. defined by a continuous spectrum of Fourier modes with random amplitudes). In this case, the phenomenon of localisation is known to take place: waves are not periodic but localised in space, with exponentially decaying tails (Molchanov & Piterbarg 1990; Sengupta, Piterbarg & Reznik 1992; Klyatskin 1996). The averaged equations, however, do not capture this phenomenon and predict exactly periodic waves whether the topography is periodic or random. This is simply because, for Rossby waves over rough topography, the characteristic scale of the localisation (the so-called localisation length) is much larger than the scale of validity of the averaged equations. In effect, the averaged equations can only describe the spatially oscillating part of the waves while neglecting the large-scale modulation that localisation induces. (See Molchanov (1991) for a general discussion of the relation between localisation and averaging, or homogenisation.)

2. Homogeneous quasi-geostrophic dynamics

2.1. *Averaged equations*

We start by considering the homogeneous quasi-geostrophic equation

$$\partial_t \left(\nabla^2\psi - \lambda^2\psi\right) + \boldsymbol{\beta}\cdot\nabla\psi + \frac{f}{H}\partial(\psi, h) + \partial(\psi, \nabla^2\psi) = 0, \tag{2.1}$$

where ψ is the streamfunction, λ^{-1} the radius of deformation, H the mean depth, h the topography height, and $\partial(\cdot,\cdot)$ the Jacobian operator. We employ the standard Cartesian β-plane but with a rotated coordinate system (x, y) and write the planetary vorticity as $f - \beta_y x + \beta_x y$, where f and $\boldsymbol{\beta} = (\beta_x, \beta_y)$ are constant.

We investigate large-scale motion over a one-dimensional, random, rough topography. Introducing $\epsilon \ll 1$ as the small ratio of the scale of the motion to the scale of the topography, the topographic height is written as

$$h(\epsilon^{-1}y) = h(Y), \quad \text{where} \quad Y = \epsilon^{-1}y. \tag{2.2}$$

It is a periodic or stationary random function with bounded derivatives, zero mean

$$\langle h\rangle = 0,$$

where $\langle\cdot\rangle$ denotes period or ensemble average, and with fixed correlation function $C(\eta)$ defined by

$$C(\eta) = \langle h(Y+\eta)h(Y)\rangle.$$

In the random case it is useful to introduce the Fourier transform $\hat{h}$ of h which we define by

$$h(Y) = \int \mathrm{e}^{\mathrm{i}kY}\hat{h}(k)\,\mathrm{d}k.$$

(Here and in the rest of the paper, integrals with unspecified bounds are understood to have $(-\infty, +\infty)$ as integration range.) The values $\hat{h}(k)$ can be taken as independent Gaussian variables, with

$$\langle\hat{h}(k)\hat{h}(l)\rangle = \hat{C}(k)\delta(k+l), \tag{2.3}$$

where the power spectrum $\hat{C}(k)$ is the Fourier transform of the correlation function:

$$C(\eta) = \int \mathrm{e}^{\mathrm{i}k\eta}\hat{C}(k)\,\mathrm{d}k.$$

The scaling of the topography which we use corresponds formally to a height field $h = O(1)$ and thus to a slope $\nabla h = O(\epsilon^{-1})$. This scaling, also used by Rhines & Bretherton (1973), Volosov (1976*a*,*b*,*c*) and others, is more precisely defined by introducing an inverse time scale (or typical frequency) for the large-scale motion, σ say, which is given by $|\boldsymbol{\beta}|L$ if the dynamics is dominated by linear Rossby waves but may be controlled by the topography. The relevant dimensionless assumption on the topography height then reads

$$\frac{h}{H} \sim \frac{\sigma}{f}$$

and can be recognised as the usual quasi-geostrophic scaling, with σ/f as a Rossby number. Since we are interested in nonlinear effects, it is important to determine the amplitude of the motion. It turns out that the suitable scaling, giving rise to leading-order nonlinearity, corresponds formally to $\psi = O(\epsilon)$ or, more precisely, to

$$\frac{\psi}{\sigma L} \sim \epsilon L = L_{\mathrm{t}},$$

a condition that can be identified as the requirement that the typical horizontal displacements of fluid particles be of the order of the scale of the topography L_{t}.

Adopting this scaling, we expand the streamfunction as

$$\psi = \epsilon\psi^{(0)} + \epsilon^2\psi^{(1)} + \cdots \tag{2.4}$$

and employ a multiple-scale technique to derive an averaged evolution from (2.1). Performing the substitution $\partial_y \to \epsilon^{-1}\partial_Y + \partial_y$ in that equation, we find at leading order, i.e. at $O(\epsilon^{-1})$,

$$\partial_t\partial^2_{YY}\psi^{(0)} + \partial_x\psi^{(0)}\partial^3_{YYY}\psi^{(0)} - \partial_Y\psi^{(0)}\partial^3_{xYY}\psi^{(0)} = 0.$$

We are interested in the situation where the leading-order motion depends on the large-scale coordinates only; therefore we select the solution $\psi^{(0)} = \psi^{(0)}(\mathbf{x}, t)$. At $O(1)$ we find

$$\partial_t\partial^2_{YY}\psi^{(1)} + \partial_x\psi^{(0)}\partial^3_{YYY}\psi^{(1)} + \frac{f}{H}h'\partial_x\psi^{(0)} = 0,$$

where $h' = \mathrm{d}h/\mathrm{d}Y$. The general solution is given by

$$\partial^2_{YY}\psi^{(1)} = \frac{f}{H}\left[g(x, Y - \eta) - h(Y)\right],$$

where $g(x, Y)$ is an arbitrary function, and where $\eta(\mathbf{x}, t)$ satisfies

$$\partial_t\eta = \partial_x\psi^{(0)} \tag{2.5}$$

and is the leading-order horizontal displacement of fluid particle across isobaths. Assuming that the small-scale motion is entirely driven by the large-scale flow, we require that $\partial^2_{YY}\psi^{(1)} = 0$ when $\eta = 0$ and find

$$\partial^2_{YY}\psi^{(1)} = \frac{f}{H}\left[h(Y - \eta) - h(Y)\right]. \tag{2.6}$$

This solution takes a natural interpretation in terms of vorticity conservation: $\partial^2_{YY}\psi^{(1)}$ is the vorticity response to the topographic stretching associated with the leading-order motion. We note that $\langle\partial^2_{YY}\psi^{(1)}\rangle = 0$ as required for $\psi^{(1)}$ to be a periodic or stationary random function as is h.

At $O(\epsilon)$ a solvability condition must be imposed; it is obtained by averaging the $O(\epsilon)$ equation. This leads to

$$\partial_t\left(\nabla^2\psi^{(0)} - \lambda^2\psi^{(1)}\right) + \boldsymbol{\beta}\cdot\nabla\psi^{(0)} + \frac{f}{H}\langle h'\partial_x\psi^{(1)}\rangle = 0. \tag{2.7}$$

The expression (2.6) for $\psi^{(1)}$ can now be used to derive a closed nonlinear evolution equation for $\psi^{(0)}$. Integrating twice by parts and introducing (2.6), the averaged term in (2.7) is written as

$$\langle h'(Y)\partial_x\psi^{(1)}\rangle = \langle\int_{Y_0}^{Y} h(Y')\,\mathrm{d}Y'\,\partial_x\partial^2_{YY}\psi^{(1)}\rangle = -\frac{f}{H}\langle\int_{Y_0}^{Y} h(Y')\,\mathrm{d}Y'\,h'(Y-\eta)\partial_x\eta\rangle,$$

where Y_0 is an arbitrary constant. Another integration by parts then gives

$$\langle h'(Y)\partial_x\psi^{(1)}\rangle = \frac{f}{H}\langle h(Y)h(Y-\eta)\rangle\partial_x\eta = \frac{f}{H}C(\eta)\partial_x\eta,$$

using the evenness of the correlation coefficient $C(\eta)$. Introducing this result into (2.7) provides the averaged, or homogenised, evolution equation for $\psi^{(0)}$. This equation is coupled with equation (2.5) for the displacement η. Omitting the superscript (0) of the streamfunction for simplicity we write these two equations as

$$\partial_t \left(\nabla^2\psi - \lambda^2\psi\right) + \boldsymbol{\beta}\cdot\nabla\psi + K(\eta)\partial_x\eta = 0, \tag{2.8}$$

$$\partial_t\eta - \partial_x\psi = 0, \tag{2.9}$$

where $K(\eta) = f^2C(\eta)/H^2$.

These equations, previously derived with $\lambda = 0$ by Volosov (1976*a,b,c*), describe the weakly nonlinear quasi-geostrophic motion over the topography $h(Y)$. The only nonlinearity appears in the correlation function $K(\eta)$; when it is neglected, i.e. when linearised equations of motion are considered, $K(\eta)$ is replaced by the constant $K(0)$ given by f^2/H^2 times the variance of the topography. In this case, and when normal modes are sought, the equations derived by Rhines & Bretherton (1973) are recovered. Similarly, the spectral equations derived by Benilov (2000) to study the effect of topography on homogeneous instability can be recovered: these equations, which include the effect of an $O(1)$ shear flow $(U(y), 0)$, are obtained when the time derivatives in (2.8)–(2.9) are replaced by $\partial_t + U\partial_x$ and the first component of $\boldsymbol{\beta}$ is replaced by $\beta_x - U''$.

Rhines & Bretherton (1973) and Benilov (2000) pointed out the analogy between quasi-geostrophic motion over one-dimensional topography with $\beta = \lambda = 0$ and two-dimensional stratified fluids (see, e.g., Gill 1982, §6.4). This is transparent from (2.8)–(2.9) when it is linearised: up to a normalisation, η plays the role of the (potential) temperature and $K(0)$ the role of the square of the Brunt-Väisälä frequency. Thus, in the same way as stratification introduces a restoring force inhibiting motion across isopycnals, topography introduces a restoring force inhibiting motion across isobaths.

The nonlinearity present in (2.8)–(2.9) is clearly associated with the topography; it dominates the advective nonlinearity $\partial(\psi, \nabla^2\psi)$ — which has order $O(\epsilon^2)$ only — and it can be interpreted as a nonlinear dispersion. Since $K(\eta)$ is expected to decrease (possibly non monotonically) from a maximum for $\eta = 0$ to zero at for $\eta \to \infty$, nonlinearity is seen to decrease the restoring force associated with the topography. The consequences of this effect for finite-amplitude wave propagation have been investigated by Volosov (1976*a,b,c*) who found periodic and solitary waves as solutions of the averaged equations (2.8)–(2.9).

The validity of the averaged equations for random topographies can be assessed by estimating the first-order term in the perturbation expansion (2.4). From (2.6) we find that it can be written as

$$\psi^{(1)} = -\frac{f}{H}\int_{Y-\eta}^{Y}\int_0^{Y'} h(Y'')\,\mathrm{d}Y'\mathrm{d}Y''.$$

With this result it is easy to show that $\langle(\psi^{(1)})^2\rangle \sim Y$ provided that $\hat{C}(0)$ is finite, i.e. the integral of $C(Y)$ is bounded as $Y \to \infty$; this implies that $\psi^{(1)}$ typically grows like $Y^{1/2}$. Therefore, over the spatial scale of interest, namely $y = O(1)$ or $Y = O(\epsilon^{-1})$, the expansion (2.4) remains well ordered, and the averaged equations (2.8)–(2.9) are valid with an $O(\epsilon^{1/2})$ error.† Phenomena not captured by these equations, such as localisation, take place over large spatial scales: $y = O(\epsilon^{-1/2})$ or larger. In fact, Molchanov & Piterbarg (1990, Eq. (10)) found for a specific random function $h(Y)$ that the localisation length scales like ϵ^{-1} (see also Sengupta, Piterbarg & Reznik 1992, Eq. (11)).

We note that the accuracy of the averaged equations is improved when $\hat{C}(0) = 0$: it is in particular easy to show that $\langle(\psi^{(1)})^2\rangle$ is bounded provided that $\hat{C}(k) = O(k^2)$

† This error estimate can be confirmed by a rigorous averaging procedure which eliminates low-order rapidly varying terms by means of near-identity transformations (cf. Arnold 1988, Ch. 5).

for $k \to 0$, suggesting that the error is at most $O(\epsilon)$. More generally, we expect the error to decrease, and correspondingly the localisation length to increase, the faster $\hat{C}(k)$ tends to zero with k. The importance of the small-k behaviour of the power spectrum $\hat{C}(k)$ for the accuracy of the averaging is quite natural since this quantity controls, in a statistical sense, the scale separation that exists between topography and large-scale motion. In this respect, it is worth pointing out that the validity of the averaging for non-degenerate random topographies hinges on the fact that the power spectrum of the random process present in the equation of motion, namely $h'(Y)$, necessarily vanishes for $k = 0$ since it is given by $k^2\hat{C}(k)$; were this not the case, the averaged equations would not be valid, and the localisation length would be $O(1)$ as has been found for different physical systems (Molchanov 1991).

A direct extension of the analysis leading to (2.8)–(2.9) to include the effect of a small-scale shear flow may be of some interest. It turns out that adding the effect of a small-amplitude zonal shear flow $(\epsilon u(Y), 0)$, where $u(Y)$ is a periodic or random function similar to $h(Y)$, leaves the analysis virtually unchanged. In particular, the averaged equations (2.8)–(2.9) continue to hold, but with a new definition for $K(\eta)$, namely

$$K(\eta) = \frac{f^2}{H^2}\langle h(Y+\eta)h(Y)\rangle - \frac{f}{H}\langle h(Y+\eta)u'(Y)\rangle.$$

Note that it is only through its correlation with the topography that the shear flow influences the averaged dynamics. In the absence of topography, the effect of the shear flow is much weaker, and a scaling very different from the one used here is required to derive averaged equations (see, e.g., Gama *et al.* (1994)).

2.2. *Hamiltonian structure*

The homogeneous quasi-geostrophic equation has a Hamiltonian, or more precisely Poisson structure, which proves useful, for instance to derive invariants and to investigate stability issues (see, e.g., Shepherd (1990)). Here we present the corresponding structure for the averaged equations (2.8)–(2.9).

Using the vorticity $\omega = \nabla^2\psi - \lambda^2\psi$ and displacement η as dynamical variables, it is easy to check that (2.8)–(2.9) can be cast in the Poisson form $\partial_t\omega = \{\omega, \mathcal{E}\}$, $\partial_t\eta = \{\eta, \mathcal{E}\}$ with a Hamiltonian given by

$$\mathcal{E} = \frac{1}{2}\iint \left[|\nabla\psi|^2 + \lambda^2\psi^2 + M(\eta)\right] \mathrm{d}x\mathrm{d}y$$

and a Poisson bracket defined for two functionals $\mathcal{F}$ and $\mathcal{G}$ by

$$\{\mathcal{F}, \mathcal{G}\} = \iint \left[\frac{\delta\mathcal{F}}{\delta\omega}\boldsymbol{\beta}\cdot\nabla\frac{\delta\mathcal{G}}{\delta\omega} + \partial_x\left(\frac{\delta\mathcal{F}}{\delta\omega}\right)\frac{\delta\mathcal{G}}{\delta\eta} - \frac{\delta\mathcal{F}}{\delta\eta}\partial_x\left(\frac{\delta\mathcal{G}}{\delta\omega}\right)\right]\mathrm{d}x\mathrm{d}y.$$

In the Hamiltonian function, $M(\eta)$ is defined by $M'' = 2K$ and can be interpreted as a potential energy associated with the topography.

Conservation laws are readily derived for the two Casimir functionals $\mathcal{C}_1$ and $\mathcal{C}_2$, given by

$$\mathcal{C}_1 = \iint \omega\,\mathrm{d}x\mathrm{d}y \quad \text{and} \quad \mathcal{C}_2 = \iint \eta\,\mathrm{d}x\mathrm{d}y,$$

as well as for the Hamiltonian $\mathcal{E}$. If $\boldsymbol{\beta} = (\beta_x, 0)$, i.e. if the isobaths are parallel to the lines of constant Coriolis parameter f so that the system is invariant under translation in the x-direction, Noether's theorem can be used to derive an additional invariant, namely the momentum

$$\mathcal{M} = -\iint \left(\eta\omega + \frac{\beta_x}{2}\eta^2 \right) \mathrm{d}x\mathrm{d}y.$$

3. Continuously stratified quasi-geostrophic dynamics

3.1. *Averaged equations*

We now consider the effect of a small-scale topography on the large-scale dynamics of a continously stratified quasi-geostrophic fluid. The quasi-geostrophic potential-vorticity conservation equation for a Boussinesq fluid can be written

$$\partial_t q + \boldsymbol{\beta} \cdot \nabla\psi + \partial(\psi, q) = 0, \quad \text{with} \quad q = \partial^2_{xx}\psi + \partial^2_{yy}\psi + \partial_z(S^{-1}\partial_z\psi). \qquad (3.1)$$

In these expressions, z is the vertical coordinate, directed upward and $S = f^{-2}N^2$, where N is the Brunt–Väisälä frequency, a function of z only. All the other quantities are defined as in the homogeneous case (see §2.1).

Equation (3.1) is supplemented by two boundary conditions at the bottom and top boundaries defined by $z = 0$ and $z = H$, respectively. These conditions express potential-temperature conservation and read

$$\partial_t\partial_z\psi + \partial(\psi, \partial_z\psi + fS\tilde{h}) = 0 \quad \text{at} \quad z = 0,$$
$$\partial_t\partial_z\psi + \partial(\psi, \partial_z\psi) = 0 \quad \text{at} \quad z = H.$$

Here $\tilde{h}$ is the topographic height which is a periodic or stationary random function of the rapid spatial coordinate $Y = \epsilon^{-1}y$, as in the previous section. For topographic effects to appear at leading order in the homogenised equations of motion, the scaling of the height must be slightly different from the one used in the homogeneous case. It turns out that a height of order $O(\epsilon^{1/2})$ is appropriate, so we let

$$\tilde{h} = \epsilon^{1/2}h(Y),$$

with $h(Y)$ given as in (2.2). In dimensional terms, this scaling can be shown to require that the typical frequency σ of the large-scale motion is related to the unscaled height $\tilde{h}$ according to

$$\frac{\tilde{h}}{H} = \epsilon^{1/2}\frac{\sigma}{f}\left(\frac{L}{L_\mathrm{i}}\right)^2,$$

with $L_\mathrm{i} = N_0H/f$, where N_0 is a typical, say bottom, value of N (N is assumed to vary by at most $O(1)$ over the depth of the ocean). Since, as discussed in the Introduction, we assume that $L/L_\mathrm{i} = O(1)$, this implies that $\tilde{h}/H = O(\epsilon^{1/2}\sigma/f)$: the relative topographic height should be smaller by a factor $\epsilon^{1/2}$ than the Rossby number. This is consistent with the quasi-geostrophic assumptions leading to (3.1).

Because the natural aspect ratio in a stratified and rotating fluid is fixed by f/N_0, the influence of the small-scale topography is confined in a shallow boundary layer of height fL_t/N_0 near $z = 0$. It is only in this boundary layer that the flow field vary on the scale of the topography or, in other words, depend on the stretched, or fast, variable Y; this is the screening effect noted by Bobrovich & Reznik (1999). Technically, this implies that the averaged evolution equations for the large-scale motion we are seeking are to be derived using a boundary-layer approach which we now detail.

Since the depth of the boundary layer is $O(\epsilon)$ in terms of the large-scale variables, we start by introducing the fast vertical variable $Z = \epsilon^{-1}z$. Away from the boundary layer,

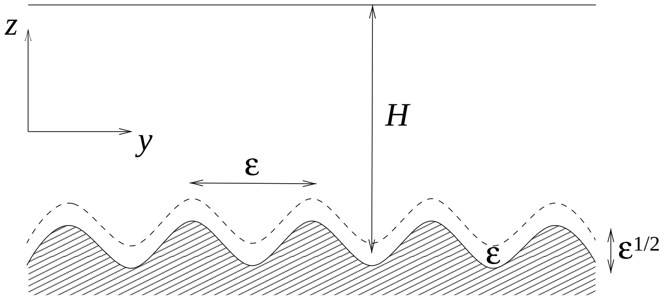

FIGURE 1. Scaling of the stratified quasi-geostrophic model: the ratio between the typical scale of motion and the horizontal scale of the topography is $\epsilon \ll 1$, and the amplitude of the topography scales like $\epsilon^{1/2}$ compared to the total depth H of the ocean. The scaling implies the existence of a boundary layer of height ϵ near the bottom boundary where the influence of topography is felt directly.

in the outer region $z = O(1)$, the streamfunction is expanded according to

$$\psi = \epsilon \left[\psi^{(0)}(\mathbf{x}, z, t) + \epsilon^{1/2} \psi^{(1)}(\mathbf{x}, z, t) + \cdots \right],$$

where $\mathbf{x} = (x, y)$ are the (slow) horizontal coordinates. The amplitude of the streamfunction is taken of order $O(\epsilon)$ as in the homogeneous case since this choice implies the appearance of a non-trivial nonlinearity at leading order. This leads to the simple leading-order equation of motion

$$\partial_t q^{(0)} + \boldsymbol{\beta} \cdot \nabla \psi^{(0)} = 0, \quad \text{with} \quad q^{(0)} = \partial_{xx}^2 \psi^{(0)} + \partial_{yy}^2 \psi^{(0)} + \partial_z (S^{-1} \partial_z \psi^{(0)}), \tag{3.2}$$

namely the linearised potential-vorticity conservation equation. The upper boundary condition is

$$\partial_t \partial_z \psi^{(0)} = 0 \quad \text{at} \quad z = H. \tag{3.3}$$

The lower boundary condition is found by considering the dynamics within the boundary layer. In this inner region, with $z = O(\epsilon)$, i.e., $Z = O(1)$, the streamfunction is expanded as

$$\psi = \epsilon \left[\psi^{(0)}(\mathbf{x}, 0, t) + \epsilon^{1/2} \phi^{(1)}(\mathbf{x}, Y, Z, t) + \epsilon \phi^{(2)}(\mathbf{x}, Y, Z, t) + \cdots \right].$$

The choice of the leading-order streamfunction is made to ensure a proper matching between the inner and outer expansion. The leading-order inner equation of motion appears at $O(\epsilon^{-1/2})$ and takes the form

$$\left(\partial_t + \partial_x \psi^{(0)} \partial_Y \right) \left(\partial_{YY}^2 \phi^{(1)} + S^{-1} \partial_{ZZ}^2 \phi^{(1)} \right) = 0.$$

The corresponding boundary condition appears at $O(\epsilon^{1/2})$ and reads

$$\left(\partial_t + \partial_x \psi^{(0)} \partial_Y \right) \partial_Z \phi^{(1)} + f S h' \partial_x \psi^{(0)} = 0 \quad \text{at} \quad Z = 0.$$

A rapidly varying solution to these last two equations should satisfy

$$\partial_{YY}^2 \phi^{(1)} + S^{-1} \partial_{ZZ}^2 \phi^{(1)} = 0,$$

and

$$\partial_Z \phi^{(1)} = f S \left[h(Y - \eta) - h(Y) \right] \quad \text{at} \quad Z = 0,$$

and $\phi^{(1)} \to 0$ as $Z \to \infty$. Here, $\eta = \eta(\mathbf{x}, t)$ is the horizontal displacement across isobaths, which obeys

$$\partial_t \eta = \partial_x \psi^{(0)} \quad \text{at} \quad z = 0. \tag{3.4}$$

Such a solution $\phi^{(1)}$ matches the outer solution which is independent of Y. It is formally given by

$$\phi^{(1)} = -fS^{1/2} \int \hat{h}(l) \frac{\mathrm{e}^{\mathrm{i}l(Y-\eta)} - \mathrm{e}^{\mathrm{i}lY}}{|l|} \mathrm{e}^{-S^{1/2}|l|Z} \, \mathrm{d}l + \psi^{(1)}(\mathbf{x}, z = 0, t). \tag{3.5}$$

At $O(1)$, the inner equation of motion reads

$$\left(\partial_t + \partial_x \psi^{(0)} \partial_Y\right) \left(\partial^2_{YY} \phi^{(2)} + S^{-1} \partial^2_{ZZ} \phi^{(2)}\right) = 0,$$

with corresponding lower boundary condition appearing at $O(\epsilon)$ in the form

$$\left(\partial_t + \partial_x \psi^{(0)} \partial_Y\right) \partial_Z \phi^{(2)} + fSh' \partial_x \phi^{(1)} = 0 \quad \text{at} \quad Z = 0.$$

Averaging these two equations leads to

$$\partial_t \langle S^{-1} \partial^2_{ZZ} \phi^{(2)} \rangle = 0,$$

and

$$\partial_t \partial_Z \langle \phi^{(2)} \rangle + fS \langle h' \partial_x \phi^{(1)} \rangle = 0 \quad \text{at} \quad Z = 0. \tag{3.6}$$

A solution for $\langle \phi^{(2)} \rangle$ matching the outer expansion is

$$\langle \phi^{(2)} \rangle (\mathbf{x}, Z, t) = Z \partial_z \psi^{(0)}(\mathbf{x}, z = 0, t) + \psi^{(2)}(\mathbf{x}, z = 0, t),$$

which obeys (3.6) provided that

$$\partial_t \partial_z \psi^{(0)} + fS \langle h' \partial_x \phi^{(1)} \rangle = 0 \quad \text{at} \quad Z = 0. \tag{3.7}$$

This furnishes the boundary condition for the leading-order outer equation (3.2). We can write (3.7) as a function of $\psi^{(0)}$ and η by using (3.5) to eliminate $\phi^{(1)}$ from the averaged term which becomes

$$\begin{aligned} \langle h' \partial_x \phi^{(1)} \rangle &= -fS^{1/2} \partial_x \eta \iint \langle \hat{h}(k) \hat{h}(l) \rangle k \, \mathrm{sgn}(l) \, \mathrm{e}^{\mathrm{i}[kY + l(Y-\eta)]} \, \mathrm{d}k \mathrm{d}l \\ &= fS^{1/2} \partial_x \eta \int \mathrm{e}^{\mathrm{i}k\eta} |k| \hat{C}(k) \, \mathrm{d}k \end{aligned}$$

using (2.3).

The averaged equations of motion, (3.2), (3.3), (3.4) and (3.7) with the superscript (0) dropped for convenience take the final form

$$\partial_t q + \boldsymbol{\beta} \cdot \nabla \psi = 0, \quad q = \partial^2_{xx} \psi + \partial^2_{yy} \psi + \partial_z (S^{-1} \partial_z \psi) \tag{3.8}$$

$$\partial_t \partial_z \psi + K(\eta) \partial_x \eta = 0 \quad \text{at} \quad z = 0, \tag{3.9}$$

$$\partial_t \eta - \partial_x \psi = 0 \quad \text{at} \quad z = 0, \tag{3.10}$$

$$\partial_t \partial_z \psi = 0 \quad \text{at} \quad z = H. \tag{3.11}$$

Here, $K(\eta)$ is the non-local, linear functional of the correlation function $C(\eta)$ defined by

$$K(\eta) = f^2 S^{3/2} \int \mathrm{e}^{\mathrm{i}k\eta} |k| \hat{C}(k) \, \mathrm{d}k = \frac{N^3}{f} \int \mathrm{e}^{\mathrm{i}k\eta} |k| \hat{C}(k) \, \mathrm{d}k.$$

These equations of motion present many similarities with equations (2.8)–(2.9) derived in the homogeneous approximation; in particular the sole nonlinearity is associated with the topography and appears in the coefficient $K(\eta)$. Note, however, that in the stratified case this coefficient is not simply the (scaled) topography correlation function but a linear transform thereof. A simple calculation shows that this linear transform relationship between $K(\eta)$ and $C(\eta)$ may also be written as

$$K(\eta) = \frac{N^3}{f}\frac{\mathrm{d}}{\mathrm{d}\eta}\mathcal{H}[C](\eta),$$

where $\mathcal{H}$ denotes the Hilbert transform, defined for any function $f(x)$ by the Cauchy-principal-value integral

$$\mathcal{H}[f](x) = \frac{1}{\pi}\int \frac{f(x')}{x-x'}\mathrm{d}x'.$$

For later use we note that for a sinusoidal topography given by

$$h(Y) = h_0 \sin(pY), \tag{3.12}$$

the correlation function is $C(\eta) = h_0^2\cos(p\eta)/2$ and

$$K(\eta) = \frac{\mathrm{d}}{\mathrm{d}\eta}\left[\frac{h_0^2 N^3}{2f}\sin(p\eta)\right] = \frac{h_0^2N^3p}{2f}\cos(p\eta). \tag{3.13}$$

As in the homogeneous case, it is important to assess the validity of the multiple-scale approach leading to the averaged equations (3.8)–(3.11). This requires examining the condition under which $\phi^{(1)}$, given by (3.5), is bounded. A direct calculation from (3.5) gives

$$\langle(\phi^{(1)})^2\rangle = 2f^2S\int[1-\cos(k\eta)]k^{-2}\hat{C}(k)\mathrm{e}^{-2S^{1/2}|k|z}\,\mathrm{d}k + \langle(\psi^{(1)}(\mathbf{x}, z=0,t))^2\rangle.$$

We can assume that $\psi^{(1)}(\mathbf{x}, z = 0, t)$ is bounded since it is determined by a system of equations analogous to that for $\psi^{(0)}$. Thus $\phi^{(1)}$ appears to be bounded provided that $\hat{C}(k) = O(1)$ as $k \to 0$. This is a more favourable situation than in the homogeneous case discussed in §2.1 (where $\psi^{(1)}$ increased like $Y^{1/2}$) so that localisation, if it occurs in stratified fluids, is likely to have a characteristic length that exceeds the $O(\epsilon^{-1})$ found for homogeneous fluids by Molchanov & Piterbarg (1990).

3.2. *Hamiltonian structure*

The averaged equations (3.8)–(3.9) have a Hamiltonian structure which takes a convenient form when q, η and $\theta := S^{-1}\partial_z\psi|_{z=0}$ are taken dynamical variables. The Hamiltonian is given by

$$\mathcal{E} = \frac{1}{2}\iiint\left[(\partial_x\psi)^2 + (\partial_y\psi)^2 + S^{-1}(\partial_z\psi)^2\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z + \frac{S^{-1}(0)}{2}\iint_{z=0} M(\eta)\,\mathrm{d}x\mathrm{d}y,$$

where the streamfunction ψ is functionally related to q and θ according to

$$\partial^2_{xx}\psi + \partial^2_{yy}\psi + \partial_z(S^{-1}\partial_z\psi) = q, \quad S^{-1}\partial_z\psi|_{z=0} = \theta, \quad \partial_z\psi|_{z=H} = 0,$$

and where, as before, $M'' = 2K$. The Poisson bracket takes the form

$$\{\mathcal{F},\mathcal{G}\} = \iiint \frac{\delta\mathcal{F}}{\delta q}\boldsymbol{\beta}\cdot\nabla\frac{\delta\mathcal{G}}{\delta q}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z + \iint_{z=0}\left[\partial_x\left(\frac{\delta\mathcal{F}}{\delta\theta}\right)\frac{\delta\mathcal{G}}{\delta\eta} - \frac{\delta\mathcal{F}}{\delta\eta}\partial_x\left(\frac{\delta\mathcal{G}}{\delta\theta}\right)\right]\mathrm{d}x\mathrm{d}y. \tag{3.14}$$

Three Casimir invariants are readily found; they are given by

$$\mathcal{C}_1 = \iiint q\,\mathrm{d}x\mathrm{d}y\mathrm{d}z, \quad \mathcal{C}_2 = \iint_{z=0}\theta\,\mathrm{d}x\mathrm{d}y \quad \text{and} \quad \mathcal{C}_3 = \iint_{z=0}\eta\,\mathrm{d}x\mathrm{d}y.$$

When $\boldsymbol{\beta} = (\beta_x, 0)$ Noether's theorem yields the momentum invariant

$$\mathcal{M} = \iiint \frac{q^2}{2\beta_x} \mathrm{d}x\mathrm{d}y\mathrm{d}z - \iint_{z=0} \theta\eta \, \mathrm{d}x\mathrm{d}y.$$

If $\beta_x = 0$, the volume integral should be omitted.

3.3. *Surface quasi-geostrophic dynamics*

When $\boldsymbol{\beta} = 0$, the interior potential-vorticity dynamics governed by (3.8) becomes trivial. Assuming that $q = 0$ and $\partial_z \psi|_{z=H} = 0$, one can derive closed evolution equations for the dynamics of the bottom potential temperature. These equations can be viewed as resulting from the averaging over small-scale topography of the surface quasi-geostrophic model studied, among others, by Held *et al.* (1995). The two conditions $q = 0$ and $\partial_z \psi|_{z=H} = 0$ provide in fact a linear relationship between ψ and $\partial_z \psi$ at $z = 0$. This relation is best expressed in terms of Fourier transforms: let $\hat{\psi}$ be the Fourier transform of $\psi|_{z=0}$, with

$$\psi(x, y, 0, t) = \iint \mathrm{e}^{\mathrm{i}(kx+ly)} \hat{\psi}(k, l, t) \, \mathrm{d}k\mathrm{d}l,$$

and, similarly, let $\widehat{\partial_z \psi}$ be the Fourier transform of $\partial_z \psi|_{z=0}$. Assuming that S is constant for simplicity, it is easy to establish that

$$\widehat{\partial_z \psi} = -mS^{1/2} \tanh\left(mS^{1/2} H\right) \hat{\psi}, \quad \text{with} \;\; m = (k^2 + l^2)^{1/2}. \tag{3.15}$$

This relation between Fourier transforms implies the existence of a linear, self-adjoint pseudo-differential operator $\mathcal{L}$ such that

$$\partial_z \psi|_{z=0} = \mathcal{L} \; \psi|_{z=0} \, .$$

(An explicit form of this operator as a convolution can be obtained from (3.15) using the convolution theorem.) Using this relationship, the dynamics may be formulated as a closed system for variables defined on the bottom boundary, namely

$$\partial_t \mathcal{L}\psi + K(\eta)\partial_x \eta = 0, \quad \partial_t \eta - \partial_x \psi = 0, \tag{3.16}$$

where ψ now denotes $\psi|_{z=0}$. This system is formally analogous to that derived in the homogeneous case (with $\beta = \lambda = 0$), with the important difference that the Laplacian operator relating vorticity and streamfunction in the homogeneous case is replaced here by the pseudo-differential operator $\mathcal{L}$ relating the potential temperature θ to the streamfunction.

The surface quasi-geostrophic model (3.16) obviously admits a Hamiltonian structure; with dynamical variables $\theta = S^{-1} \, \partial_z \psi|_{z=0} = S^{-1} \mathcal{L}\psi$ and η, the Hamiltonian is given by

$$\mathcal{E} = \frac{1}{2} \iint \left[-S\theta\mathcal{L}^{-1}\theta + S^{-1} M(\eta) \right] \mathrm{d}x\mathrm{d}y,$$

while the Poisson bracket is given by the surface term in (3.14).

4. Waves in continuously stratified fluids

We can examine the finite-amplitude travelling waves which exist in the presence of topography and β-effect. These waves can be regarded as the stratified counterparts to those found by Volosov (1976*a*,*b*,*c*) in the homogeneous case. Consider a streamfunction and displacement of the form $\psi = \psi(x + \gamma y - ct, z)$ and $\eta = \eta(x + \gamma y - ct)$. With constant

S, the averaged equations (3.8)–(3.11) become

$$
\begin{aligned}
-c\left[(1+\gamma^2)\partial^3_{xxx}\psi + S^{-1}\partial^3_{xzz}\psi\right] + \beta\partial_x\psi = 0,& \\
-c\partial^2_{xz}\psi + K(\eta)\partial_x\eta = 0 \quad \text{at} \quad z = 0,& \\
-c\partial_x\eta - \partial_x\psi = 0 \quad \text{at} \quad z = 0,& \\
-c\partial^2_{xz}\psi = 0 \quad \text{at} \quad z = H,&
\end{aligned}
$$

where $\beta = \beta_x + \gamma\beta_y$. From the interior vorticity equation and surface boundary condition, both of which are linear, it is possible to derive a linear relationship between $\psi|_{z=0}$ and $\partial_z\psi|_{z=0}$ similar to that derived in §3.3 under the assumption that $\beta = 0$. In terms of the (one-dimensional) Fourier transform $\hat{\psi}$, with

$$\psi(x,0) = \int \mathrm{e}^{\mathrm{i}kx}\hat{\psi}(k)\,\mathrm{d}k,$$

we find that $\widehat{\partial_z\psi}$, the Fourier transform of $\partial_z\psi|_{z=0}$, satisfies

$$\widehat{\partial_z\psi} = -mS^{1/2}\tanh\left(mS^{1/2}H\right)\hat{\psi}, \quad \text{with} \quad m = \left[k^2(1+\gamma^2) + \frac{\beta}{c}\right]^{1/2}. \tag{4.1}$$

This implies a linear relationship of the form

$$\partial_z\psi|_{z=0} = \mathcal{L}_\beta\ \psi|_{z=0}\,,$$

with a pseudo-differential operator $\mathcal{L}_\beta$ which depends parametrically on γ and c as well as on β. With this relationship, the travelling-wave problem can be formulated as a single pseudo-differential equation for $\eta(x)$: indeed, the two equations for the bottom boundary condition can be combined to find

$$c^2\partial_x\mathcal{L}_\beta\eta + K(\eta)\partial_x\eta = 0, \tag{4.2}$$

and, integrating once,

$$c^2\mathcal{L}_\beta\eta + L(\eta) = \text{const.},$$

where $L(\eta) = N^3\mathcal{H}[C](\eta)/f$ is an indefinite integral of $K(\eta)$.

4.1. *Linear waves*

The dispersion relation for linear waves is deduced directly from equation (4.2). The linearisation is carried out by replacing the function $K(\eta)$ by the constant $K(0)$. For a displacement field η proportional to $\exp[\mathrm{i}(kx + ly - ct)]$ we find from (4.1) and (4.2) the dispersion relation

$$c^2(k^2 + l^2 + \beta/c)^{1/2}S^{1/2}\tanh\left[\left(k^2 + l^2 + \beta/c\right)^{1/2}S^{1/2}H\right] = K(0). \tag{4.3}$$

This transcendental expression relates the wave phase velocity c to the wavevector (k, l). It is equivalent to that obtained by Bobrovich & Reznik (1999, Eq. (3.21*b*)), under our additional assumption $L_\mathrm{t}/L_\mathrm{i} \ll 1$ which is necessary for a formal asymptotic derivation.

Bobrovich & Reznik (1999) discuss the various solutions to the dispersion relation and identify homogeneous and baroclinic Rossby modes and a topographic mode. We refer the reader to this paper for details; here, as an illustration, we take $\beta = 0$ and $H \to \infty$ to consider only the purely topographic modes whose nonlinear extension is discussed in the next section. In this case, the dispersion relation reduces to

$$c^2(k^2 + l^2)^{1/2}S^{1/2} = K(0).$$

Specialising to the case of a sinusoidal topography, the frequency can be expressed as

$$\omega^2 = \frac{N^2 h_0^2 p}{2} \frac{k^2}{(k^2+l^2)^{1/2}} \tag{4.4}$$

using (3.13). This dispersion relation is quadratic in ω: two topographic modes with opposite directions of propagation are supported by the topography, as is the case in the homogeneous model. The dependence of the frequency on the wavevector (k, l) is however different from that found in the homogeneous model; in particular the frequency is seen here to be scale dependent. Perhaps surprisingly, the frequency does not depend on the Coriolis parameter f, although rotation is essential for the existence of the topographic waves. Note however that for an ocean of finite depth $H < \infty$, a dependence of the frequency on f appears through the argument of the hyperbolic tangent in (4.3).

It is worth emphasising that the vertical structure of the waves is of the form

$$\exp\left[-N\left(k^2+l^2\right)^{1/2} z/f\right],$$

so that they are not particularly confined near the bottom boundary when the wavenumbers k and l are small; the screening effect, with localisation within a shallow, $O(\epsilon)$ bottom boundary layer, concerns thus only to the direct, small-scale response to the topography. The topography, has an indirect, large-scale response in the form of topographic waves which are not confined in a boundary layer but have an $O(1)$ vertical extent.

The linear dispersion relation (4.4) has a simple nonlinear generalisation for finite-amplitude waves which we now discuss.

4.2. *Nonlinear waves*

To examine nonlinear waves, we directly concentrate on the case $\beta = 0$. Travelling waves satisfy the equation

$$c^2 \mathcal{L}_0 \eta + L(\eta) = \text{const.}, \tag{4.5}$$

where $\mathcal{L}_0$ is obtained from (4.1) by restricting to $\beta = 0$. (Alternatively, this equation can be obtained by introducing travelling-wave solutions into the averaged surface quasi-geostrophic equations (3.16).) Simple solutions to this equation can be found in the limit $H \to \infty$, when

$$\mathcal{L}_0 \to -(1+\gamma^2)^{1/2} S^{1/2} \frac{\mathrm{d}}{\mathrm{d}x} \mathcal{H},$$

In this case, (4.5) may be written

$$c^2 (1+\gamma^2)^{1/2} S^{1/2} \mathcal{H}[\eta'](x) = \frac{N^3}{f} \mathcal{H}[C](\eta(x)) + \text{const.},$$

where $\eta' = \mathrm{d}\eta/\mathrm{d}x$. For a sinusoidal topography (3.12), and with a vanishing constant, this equation reduces to

$$\mathcal{H}[\eta'](x) = \frac{N^2 h_0^2}{2c^2(1+\gamma^2)^{1/2}} \sin\left[p\eta(x)\right], \tag{4.6}$$

where we used the form (3.13) for $K(\eta)$. This can be recognized as the Peierls–Naborro equation obtained in cristal-dislocation theory (Peierls 1940; Nabarro 1947). This observation allows solutions to be derived from the work of Toland (1997): up to translation and addition of integer multiples of 2π, all periodic solutions belong to a unique family parameterised by a parameter $0 < \Gamma \le 1$. These solutions take the form

$$\eta(x) = \frac{2}{p}\left\{\tan^{-1}\left[\Gamma^{-1}\tan(kx/2)\right] - \tan^{-1}\left[\Gamma \tan(kx/2)\right]\right\}$$

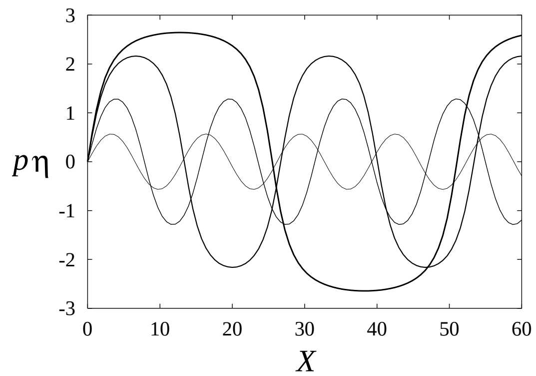

FIGURE 2. Periodic solutions of the Peierls–Nabarro equations for nonlinear travelling waves in stratified quasi-geostrophic flow over the sinusoidal topography $h_0 \sin(pY)$. The dimensionless displacement across topography is plotted against the dimensionless spatial coordinate $X = N^2 h_0^2 px/[c^2(1+\gamma^2)^{1/2}]$ for $\Gamma = 0.125, 0.25, 0.5$ and 0.75. Increasing values of Γ correspond to decreasing amplitudes, wavelengths and line widths.

$$= \frac{2}{p} \tan^{-1}\left[\frac{1}{2}\left(\Gamma^{-1} - \Gamma\right)\sin(kx)\right].$$

where k is related to Γ and c according to

$$k = \frac{N^2 h_0^2 p}{c^2(1+\gamma^2)^{1/2}} \frac{\Gamma}{1+\Gamma^2}. \tag{4.7}$$

The nonlinear dispersion relation connecting phase velocity c, wavenumber k and wave amplitude is deduced by noting that crests and troughs correspond to $x = \pm\pi/(2k)$, so that the Γ determines the amplitude according to

$$A = \frac{4}{p}\left(\tan^{-1}\Gamma^{-1} - \tan^{-1}\Gamma\right).$$

This shows, in particular, that $pA < 2\pi$. This relationship can be inverted as

$$\Gamma = \sec(pA/4) - \tan(pA/4).$$

Introducing this result into (4.7) leads to the dispersion relation in the form

$$\omega^2 = \frac{N^2 h_0^2 p}{2} \frac{k}{(1+\gamma^2)^{1/2}} \cos(pA/4)$$

which directly extends the linear result (4.4). Nonlinearity is seen to lead to a decrease in the frequency as the wave amplitude increases; it also results in a change in the wave form, as illustrated in figure 2.

The limit $pA \to 2\pi$, i.e. $\Gamma \to 0$, is an interesting one: the periodic solution tends to the kink, or front, solution

$$\eta = \frac{2}{p} \tan^{-1}\left(\frac{N^2 h^2 p}{c^2(1+\gamma^2)^{1/2}} x\right)$$

for the displacement. The velocity field associated with this displacement has the form of a solitary wave. This is similar to the results obtained by Volosov (1976*a,b,c*) for waves in a homogeneous fluid.

5. Eady's model of baroclinic instability

It is interesting to examine how large-scale baroclinic instability is affected by the presence of a (one-dimensional) small-scale topography. This has recently been considered by Benilov (2001) who extended Phillips' two-layer model of baroclinic instability to include the effect of topography. Here we use the averaged equations (3.8)–(3.11) to study baroclinic instability in a stratified fluid with topography. For simplicity we focus on extending the linear stability analysis of Eady's model; since this entails only minor modifications to the standard treatment, we only sketch the derivation and refer to Pedlosky (1987, p. 523) for details.

Eady's model considers the stability of a vertically sheared flow $(\Lambda z, 0, 0)$, where Λ is the shear, with $\beta = 0$. The interior vorticity q is assumed to vanish, so that the dynamics is governed by the surface quasi-geostrophic equations (3.9)–(3.11). Due to the basic flow, these are somewhat modified and take the form

$$\partial_t \partial_z \psi - \Lambda \partial_x \psi + K(\eta) \partial_x \eta = 0 \quad \text{at} \quad z = 0, \tag{5.1}$$

$$\partial_t \eta - \partial_x \psi = 0 \quad \text{at} \quad z = 0, \tag{5.2}$$

$$(\partial_t + \Lambda H \partial_x) \, \partial_z \psi - \Lambda \partial_x \psi = 0 \quad \text{at} \quad z = H. \tag{5.3}$$

Compared to (3.9)–(3.11), (5.1)–(5.3) contain additional terms which are associated with the mean-flow advection on the upper boundary and with the presence of a basic potential-temperature gradient induced by the shear on both the top and bottom boundaries.

(Note that we have assumed that the basic-flow velocity is in the x-direction, i.e. parallel to the isobaths. This makes the inclusion of the basic flow in the averaged equations straightforward because all the terms associated with the mean flow involve only (slow) x-derivatives, so that the equations governing $\phi^{(1)}$ remain essentially unchanged. In contrast, a different orientation of the basic flow introduces fast and slow derivatives in the y-direction and modifies the equations for $\phi^{(1)}$. Physically this follows from the non-zero potential-temperature gradient that is imposed in the x-direction by thermal-wind balance and from its advection by the $O(\epsilon^{1/2})$ velocity $\partial_Y \phi^{(1)}$.)

To study the spectral stability of the Eady basic flow, we linearise (5.1)–(5.3) by replacing $K(\eta)$ by $K(0)$ and consider normal-mode solutions. The constraint $q = 0$ imposes the form

$$\psi = \left[A \cosh \left(m S^{1/2} z \right) + B \sinh \left(m S^{1/2} z \right) \right] \mathrm{e}^{\mathrm{i}(kx + ly - \omega t)},$$

where A and B are two constants, k and l are the horizontal wavenumbers, and $m = \sqrt{k^2 + l^2}$. Introducing this form into (5.1)–(5.3) and eliminating η leads to a homogeneous linear system for A and B. Non-trivial solutions are found provided that the dispersion relation between the complex frequency ω and the wavenumbers (k, l) is satisfied. Introducing the parameters

$$c = \frac{\omega}{\Lambda H k}, \quad \mu = m S^{1/2} H \quad \text{and} \quad \kappa = \frac{K(0)}{\Lambda^2 H},$$

the dispersion relation can be written in dimensionless form as

$$c^2 - c + \left(\frac{\coth \mu}{\mu} - \frac{1}{\mu^2} \right) \left(1 + \frac{\kappa}{c} \right) - \frac{\coth \mu}{\mu} \kappa = 0.$$

In the absence of topography, $\kappa = 0$ and the quadratic equation for the dimensionless phase velocity c found in the standard Eady model is recovered (see Pedlosky 1987, p.

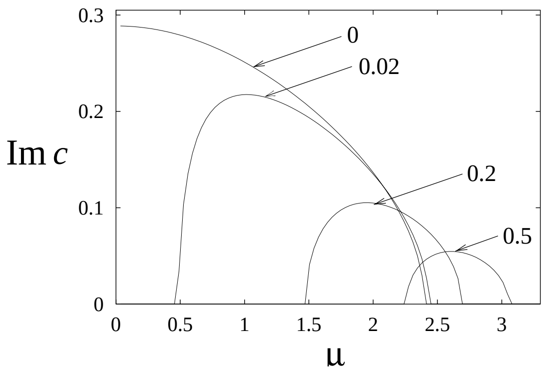

FIGURE 3. Imaginary part of the phase velocity c of unstable modes as a function of the horizontal wavenumber μ in Eady's model of baroclinic instability with small-scale topography for the topographic parameter $\kappa = 0, 0.02, 0.2$ and 0.5.

525). When $\kappa \neq 0$, the dispersion relation is cubic and can be easily solved, leading to three values of c, with at most two complex conjugate values associated with instability.

Figure 3 displays the imaginary part of c of potentially unstable modes as a function of μ for several values of the topographic parameter κ. It shows that the presence of topography has a stabilising effect on the flow, reducing $\mathrm{Im}\, c$ as well as the growth rate $k\mathrm{Im}\, c$. At the same time, the range of unstable wavenumbers is shifted toward small scales. Qualitatively similar conclusions were drawn by Benilov (2001) in his study of baroclinic instability in a two-layer model.

6. Discussion

This paper employs a multiple-scale approach to derive averaged evolution equations describing quasi-geostrophic motion over a small-scale one-dimensional topography. Small- but finite-amplitude motion is considered, leading to averaged equations that are nonlinear, with the nonlinearity entering only through the topographic term. The averaged equations are given by (2.8)–(2.9) in the case of a homogeneous quasi-geostrophic fluid, already treated by Volosov (1976*c*,*a*,*b*), and by (3.8)–(3.11) in the case of a continuously stratified quasi-geostrophic fluid. The latter case can be regarded as providing a nonlinear extension to the linear results of Bobrovich & Reznik (1999) and as complementing the nonlinear results of Volosov & Zhdanov (1983) obtained for a different scaling regime.

The averaged equations derived in this paper and in its predecessors provide what may be interpreted as a parameterisation of the effect of small-scale topography on large-scale flows. This parameterisation has the advantage that it is derived deductively using an asymptotic approach rather than heuristically; however the simplifying assumptions which make such an asymptotic derivation possible (weak nonlinearity in particular) suggest that the parameterisation is not adapted for practical implementation in numerical ocean models. It is nevertheless interesting to note some of the features of the averaged equations which one may wish to take into account in the design of parameterisations. Concentrating here on the continuously stratified case, we note that: (i) the small-scale topography affects only the bottom boundary condition of the quasi-geostrophic model, leaving the (linearised) interior potential-vorticity equation unchanged; (ii) the modified boundary condition is second-order in time and so involves an additional dynamical variable, namely the displacement across topography; and (iii) the correlation function of the topography characterises entirely its effect on the large-scale flow.

A major limitation of the present work, which makes the results relevant only to particular oceanic areas such as ridge regions, is the assumption of a one-dimensional topography. The impact of a two-dimensional topography on linear, homogeneous quasi-geostrophic motion has recently been investigated by the author (Vanneste 2000*a*,*b*). The averaged equations derived in these papers reveal that the effect of a two-dimensional topography is significantly more complex than that of a one-dimensional topography. In particular, the averaged equations turn out to be integro-differential equations, generally not reducible to a finite set of differential equations. Also, the main topographic parameter involved in the averaged equation (in fact a function of time) cannot be expressed directly in terms of the topographic height h but is determined by solving a variable-coefficient partial-differential equation. In view of this, it is clear that a similar asymptotic treatment of the continuously stratified quasi-geostrophic model, although challenging, would be of interest.

A significant part of the present paper is devoted to the study of nonlinear travelling waves which exist because of the topography. In the continuously stratified model, these waves satisfy a pseudo-differential equation which presents a certain theoretical interest. It is shown here to be integrable in the particular case of a sinusoidal topography (and for a constant of integration taken to vanish) because it reduces to the Peierls–Nabarro equation whose solutions are explicitly known. However the sinusoidal topography is probably an oversimplification for applications and other topographies deserve attention. These lead to pseudo-differential equations similar to the Peierls–Nabaro equation (4.6), but with the sine function on the right-hand side replaced by other functions. It would interesting, if only from a theoretical viewpoint, to examine which of these functions and hence which forms (or whether all forms) of the topography lead to integrable pseudo-differential equations.

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