

Wave radiation by slow flows

The breakdown of balance in a toy model of the atmosphere

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Balance and the slow manifold

Observations: mid-latitude atmospheric time scales are

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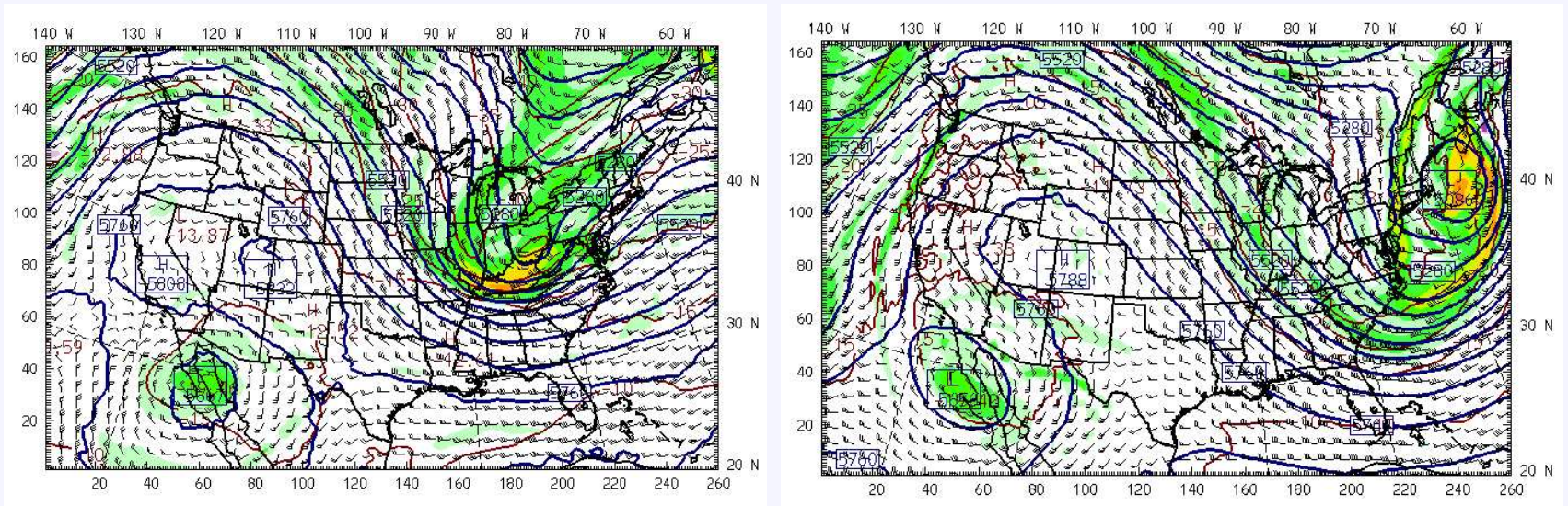
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Weather maps at 1 day interval

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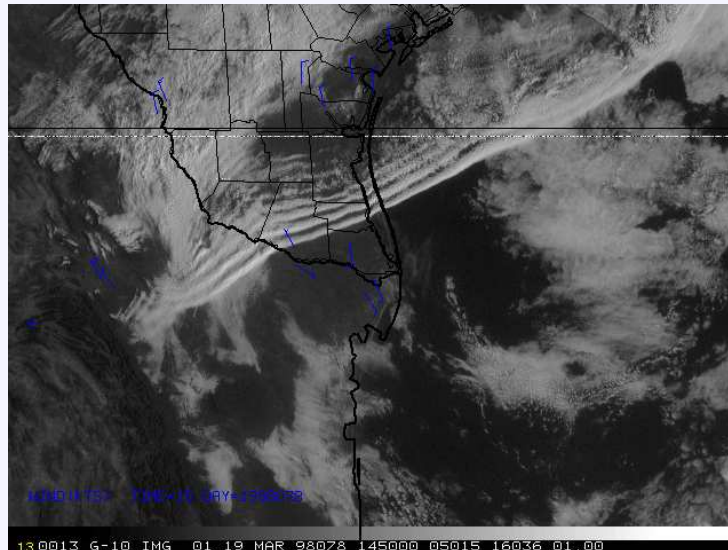
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$$\frac{\partial s}{\partial t} = N_s(s, f; \delta), \quad \frac{\partial f}{\partial t} + \mathcal{L}f = N_f(s, f; \delta)$$

with $\text{spec } \mathcal{L} = \{i\omega : \omega \in \mathbf{R}, |\omega| > \delta^{-1}\}$.

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Observations:

- fast oscillations are weak, $f \ll s$

Balance and slow manifold

Reduce the dynamics to a **slow manifold**: submanifold of state space which is approximately invariant and on which the motion is slow (van Kampen 1985, MacKay 2004).

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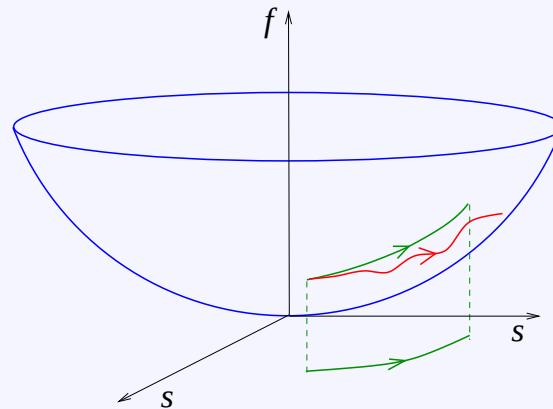
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Balance, slow manifold (continued)

Interest:

- **Balanced models:** describe dynamics on a slow manifold, with reduced dimension ($\dim s = \dim f / 2$) and large time steps,

$$\frac{\partial s}{\partial t} = N_s(s, F(f, \delta), \delta)$$

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Questions:

- Can balance be perfect? Is there an invariant slow manifold?
- If not, describe the spontaneous generation of inertia-gravity waves. What is the wave amplitude?

Lorenz's 5-component model

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Lorenz's less-famous equations:

Shallow-water on the sphere, spectral expansion, truncation to 3 modes $\rightarrow$ 9-component model

Neglect 2 IGWs $\rightarrow$ 5-component model (Lorenz 1986):

$$\dot{u} = -vw + bvy$$

$$\dot{v} = uw - buy$$

$$\dot{w} = -uv$$

$$\delta\dot{x} = -y$$

$$\delta\dot{y} = x + \delta buv$$

Slow variables: $s = (u, v, w)$ Rossby-wave triad

Fast variables: $f = (x, y)$ inertia-gravity-wave pair.

Lorenz's 5-component model (continued)

2 parameters:

- $b = f_C L / \sqrt{gH} = \text{rotational Froude number,}$
- $\epsilon = U / (f_C L) = \text{Rossby number.}$

They combine in $\delta = T_a / T_{\text{igw}} = \epsilon / \sqrt{1 + 1/b^2} = (\text{oscillation frequency})^{-1}$.

(Compare with shallow-water dispersion relation: $\omega = \epsilon^{-1} \sqrt{1 + k^2/b^2}$.)

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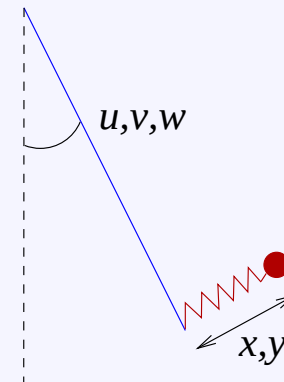
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Lorenz's model: a pendulum coupled to a stiff spring



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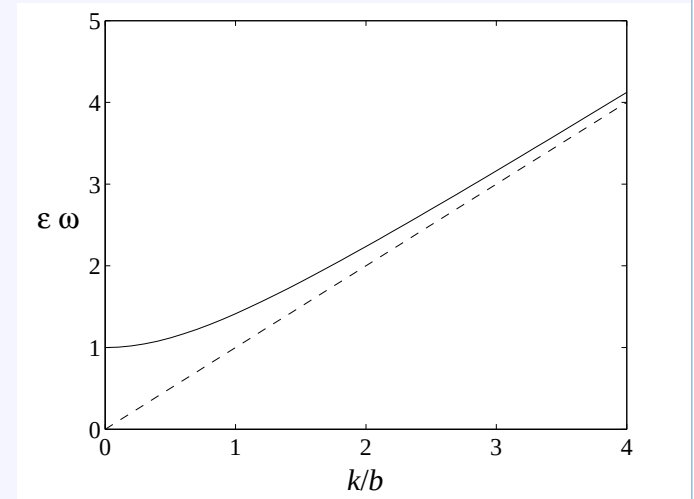
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- They can be captured by **exponential asymptotics** (V 2004, V & Yavneh 2004, Olfasdottir, Olde Daalhuis & V 2005).

Lorenz's 5-component model (continued)

The reduction from PDEs to ODEs has some drawbacks:

- ODEs cannot describe wave generation in the small-Froude number regime $b \ll 1$, $\epsilon = O(1)$.

It is caused by resonance of large-scale waves, with $k = O(b)$: **Lighthill radiation** (also gravitational waves)

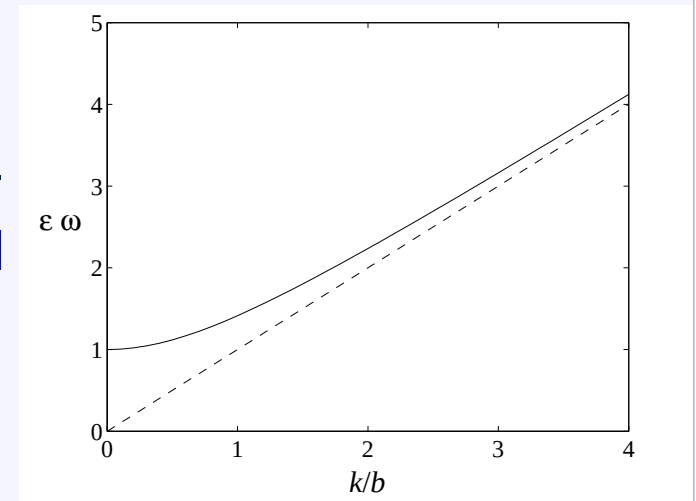


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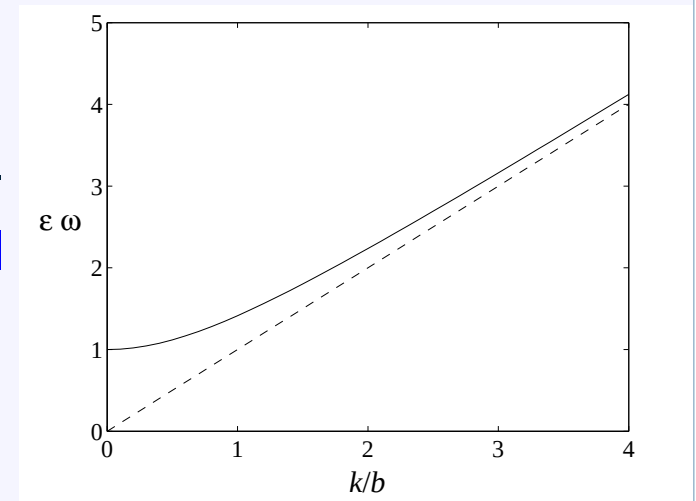
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Introduce a new toy model that retains a PDE component

Extended Lorenz model

Keep the pendulum for the slow variables, replace the oscillator by a 1D linear Klein–Gordon equation, with dispersion relation

$$\omega^2 = \epsilon^{-2}(1 + k^2/b^2).$$

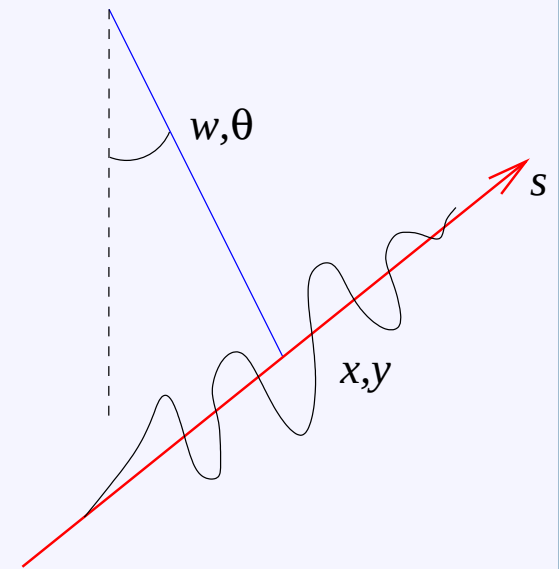
The equations of motion are:

$$\dot{\theta} = w$$

$$\dot{w} = -\sin \left[2\theta + 2\epsilon \int f(s)x(s,t) ds \right]$$

$$\epsilon x_t = -y$$

$$\epsilon y_t = x - x_{ss}/b^2 + \epsilon f(s) \sin \left[2\theta + 2\epsilon \int f(s)x(s,t) ds \right]$$



ODE/PDE coupling through the function $f(s) = \frac{a}{(2\pi)^{1/2}} \frac{d}{ds} e^{-s^2/2}$.

Extended Lorenz model

The model is Hamiltonian, with energy

$$H = \frac{w^2}{2} - \frac{1}{2} \cos \left[2\theta + 2\epsilon \int f(s)x(s,t) \, ds \right] + \frac{1}{2} \int (x_s^2/b^2 + x^2 + y^2) \, ds$$

and symplectic form $\Omega = d\theta \wedge d\omega + \epsilon dy \wedge dx$.

H escapes finite domains with flux $F = x_t x_s / b^2$.

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Use the model to study wave generation in the regimes:

- $b \ll 1$, $\epsilon = O(1)$: Lighthill-type radiation,
- $\epsilon \ll 1$, $b = O(1)$: exponentially small radiation

Small-Froude-number regime

For $b \ll 1$, long waves with $k = O(b)$ are slow, and generated resonantly.

We can use **matched asymptotics** to derive

- the form of the long waves generated (cf. Einstein 1918, Lighthill 1952),
- a **post-balanced** model: closed, dissipative model for (θ, w) that accounts for wave radiation (cf. Blanchet 2002, Ford, McIntyre & Norton 2000).

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To leading order, the waves satisfy the linear Klein–Gordon equation

$$\epsilon^2 x_{tt} - x_{SS} + x = ab^2 \epsilon \delta'(S) \sin(2\theta), \quad \text{with } S = bs$$

Dipolar generation, leading to $O(b^2)$ waves.

Small-Froude-number regime (continued)

The wave feedback appears at $O(b^3)$, leading to the **post-balanced model**

$$\begin{aligned}\dot{\theta} &= w \\ \dot{w} &= -\sin \left[2\theta - b^2 a^2 \epsilon \sin(2\theta) / \pi^{1/2} - 2b^3 a \epsilon B(t) \right]\end{aligned}$$

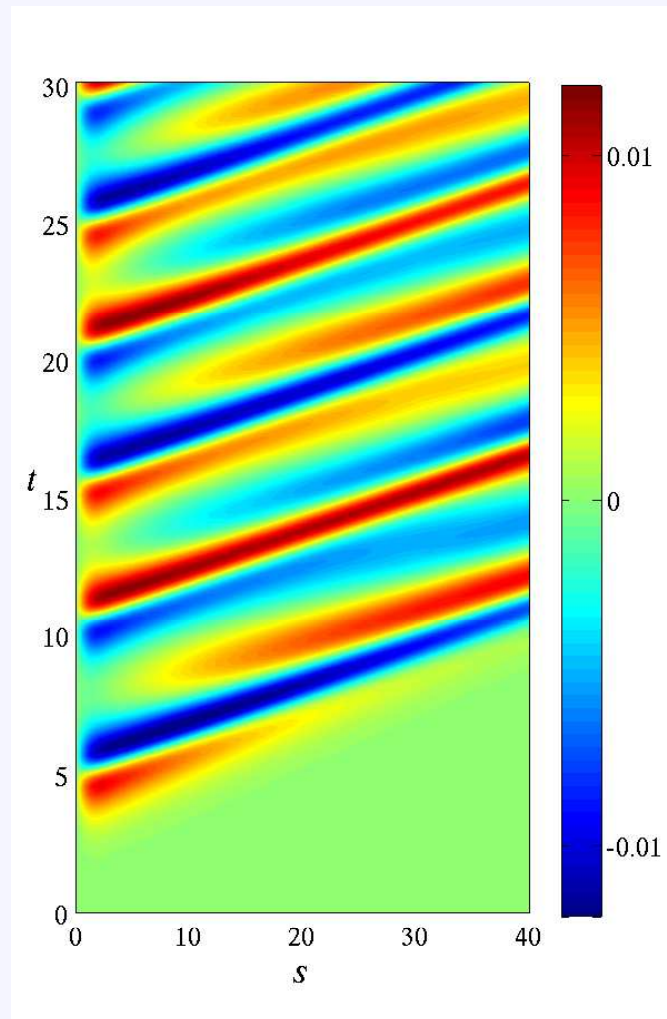
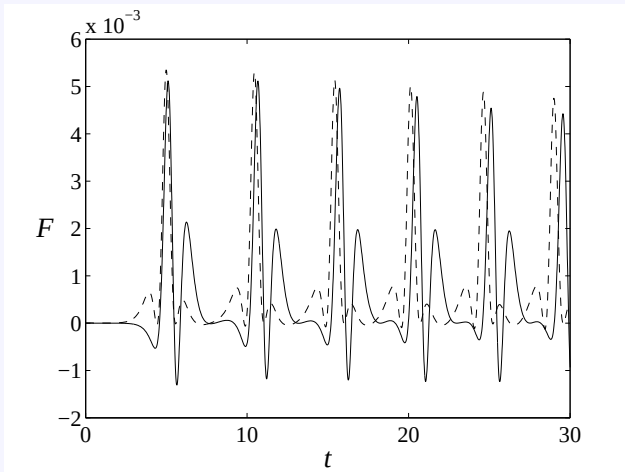
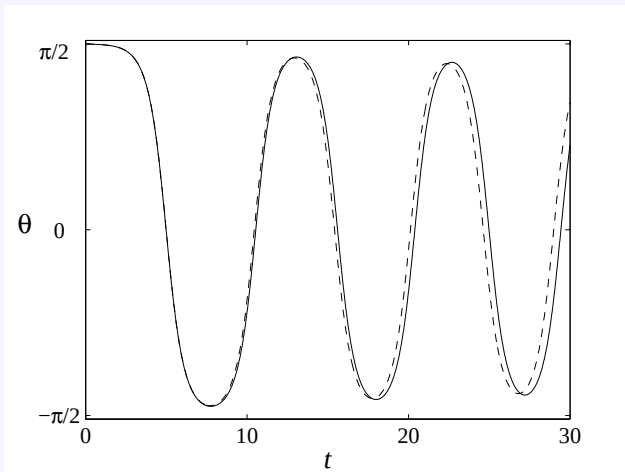
with $B(t) = -a\epsilon^2 w \cos(2\theta) - \frac{a\epsilon J_1(t/\epsilon)}{2t} \star \sin(2\theta)$.

The post-balanced model consists of **integro-differential equations**; it reduces to differential equations for non-dispersive waves (in odd dimensions).

Compare numerical solutions for initial condition on unperturbed ($b = 0$) separatrix:

$$\cos \theta = \operatorname{sech} [\sqrt{2}(t - t_0)] \quad \text{and} \quad \sin \theta = -\tanh [\sqrt{2}(t - t_0)]$$

Small-Froude-number regime (continued)



Results for $b = 0.15$, $a = \epsilon = 1$, $t_0 = 5$.

Small-Rossby-number regime

For $\epsilon \ll 1$, a **slow manifold** can be found to any accuracy $O(\epsilon^N)$ by slaving:

$$x_{\text{bal}}(s, \theta, w) = \sum_{n=0}^{\infty} \epsilon^{2n+1} x_{\text{bal}}^{(n)}(s, \theta, w),$$

with $x_{\text{bal}}^{(0)} = -\frac{ab^2 e^{b^2/2}}{4} \left[e^{bs} \operatorname{erfc} \frac{b+s}{\sqrt{2}} - e^{-bs} \operatorname{erfc} \frac{b-s}{\sqrt{2}} \right] \sin(2\theta)$, etc.

Can add wave solution:

$$x_w = \int C(k) \exp[i(ks - \omega t)] dk, \quad \text{with} \quad \omega^2 = (1 + k^2/b^2)/\epsilon^2 \gg 1.$$

but interest in **balanced solutions**: $C(k) = 0$ at $t = 0$.

No wave generation visible for $t \in \mathbb{R}$: $C(k) = o(\epsilon^N), \forall N$.

Small-Rossby-number regime (continued)

For $t \in \mathbb{C}$, near singularities of $(\theta(t), w(t))$, the slow-manifold expansion breaks, and wave generation appears at leading order.

Examine equations near singularities, and match (Kruskal–Segur): $C(k)$ jumps from 0 to exponentially small values when $t \in \mathbb{R}$ crosses Stokes lines ($\theta = 0$).

Inner problem: for $\tau = (t - t_*)/\epsilon = O(1)$, expand

$$w = \epsilon^{-1} \sum_{n=1}^{\infty} \frac{w_n}{\tau^{2n-1}}, \quad x = \epsilon^{-1} \sum_{n=1}^{\infty} \frac{X_n(s)}{\tau^{2n}}, \quad \text{etc.}$$

to find recurrence relations for w_n , $X_n(s)$, etc. For $n \gg 1$, observe that

$$\hat{X}_n \sim \frac{(-1)^{n+1} (2n-1)! \lambda(k)}{2\pi \Omega^{2n}} k e^{-k^2/2}, \quad \Omega^2 = 1 + k^2/b^2.$$

for some $\lambda(k)$, provided that $k \ll 1$.

Small-Rossby-number regime (continued)

Borel-summation of the divergent series

$$\hat{x} = \epsilon^{-1} \sum_{n=1}^{\infty} \frac{\hat{X}_n(k)}{\tau^{2n}},$$

shows that the late behaviour is associated with the wave part

$$\hat{x}_w(k, t) \sim \frac{-ik\lambda(k)}{\epsilon} e^{-\alpha\omega/\epsilon - k^2/2} \cos[\omega(t - \beta)/\epsilon], \quad k \ll 1,$$

with $\alpha = \Im t_* > 0$ and $\beta = \Re t_*$.

Inverting the Fourier transform by steepest descent gives

$$x_w(s, t) \sim (2\pi)^{1/2} b^3 \lambda(0) e^{-\alpha/\epsilon} S \Re \frac{e^{i(t-\beta)/\epsilon - b^2 S^2 / [2(\alpha - i(t-\beta))]} }{[\alpha - i(t - \beta)]^{3/2}},$$

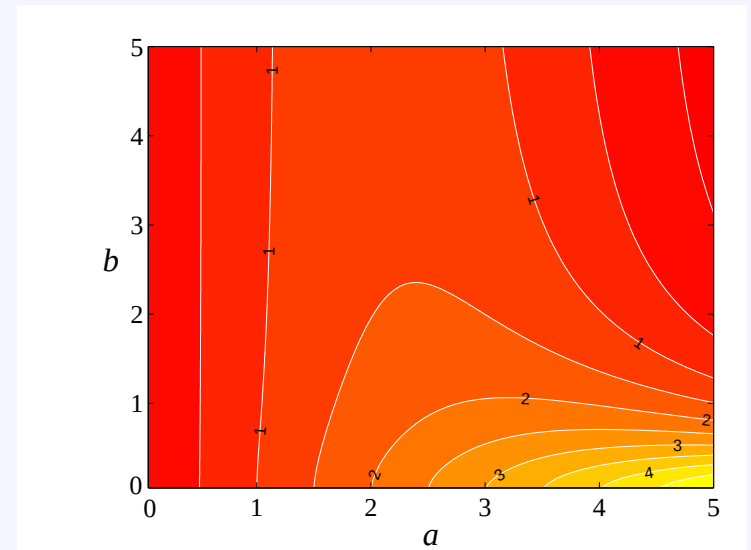
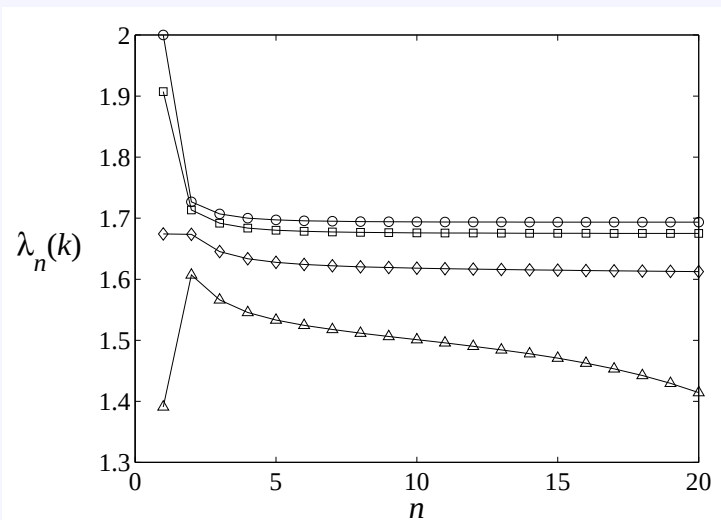
for $S = \epsilon^{1/2} s = O(1)$.

Small-Rossby-number regime (continued)

Expression for the **exponentially small waves** generated by the **balanced motion** when a Stokes line is crossed.

It depends on a single parameter $\lambda(0)$ to be determined numerically for each a and b . This is done by solving recurrence relations for $\hat{X}_n(k)$ discretised (Gauss–Hermite), and

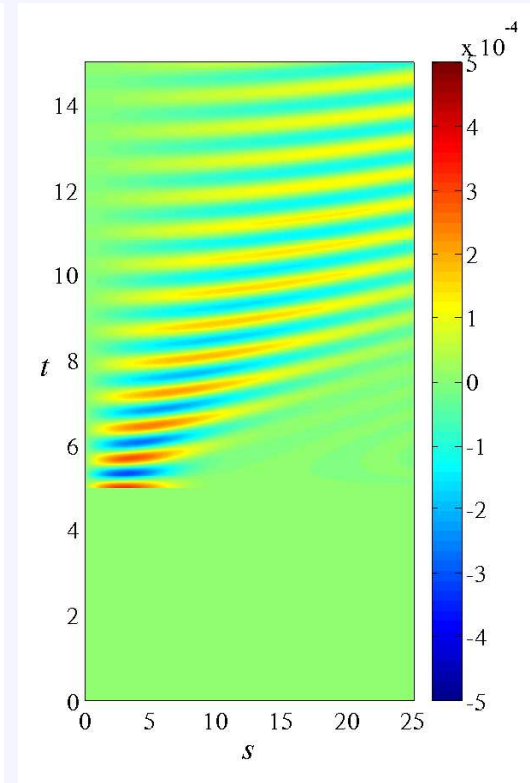
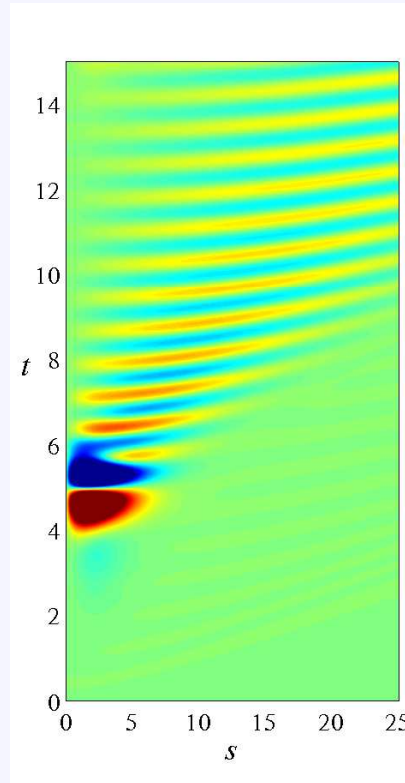
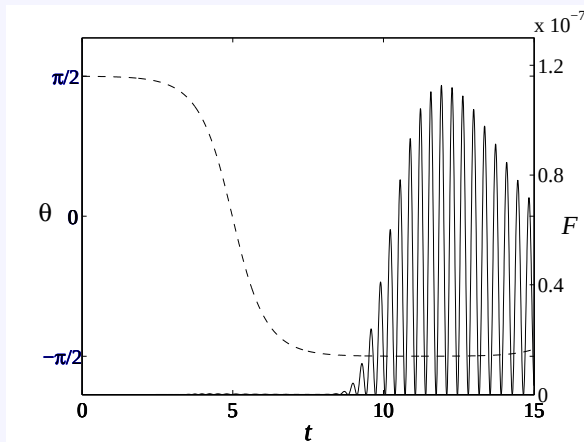
$$\frac{(-1)^{n+1} 2\pi k \Omega^{2n}}{(2n-1)!} \hat{X}_n e^{k^2/2} \rightarrow \lambda(k) \text{ as } n \rightarrow \infty.$$



For $a \ll 1$ or $b \ll 1$, $\lambda(0) \sim a$: weak coupling limits.

Small-Rossby-number regime (continued)

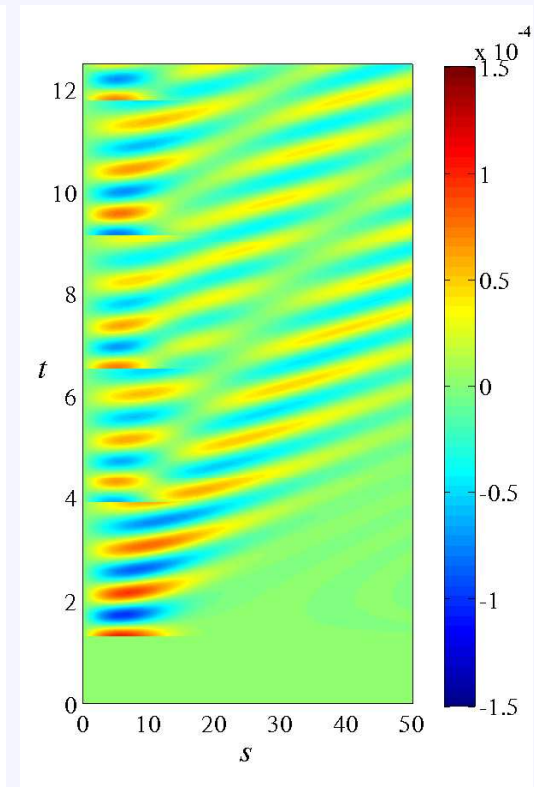
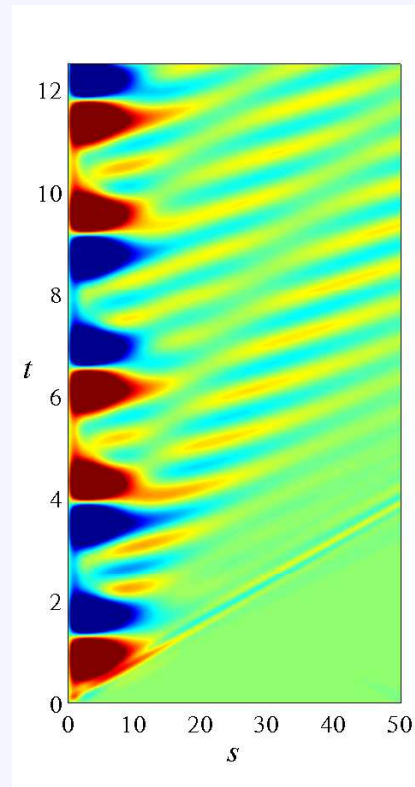
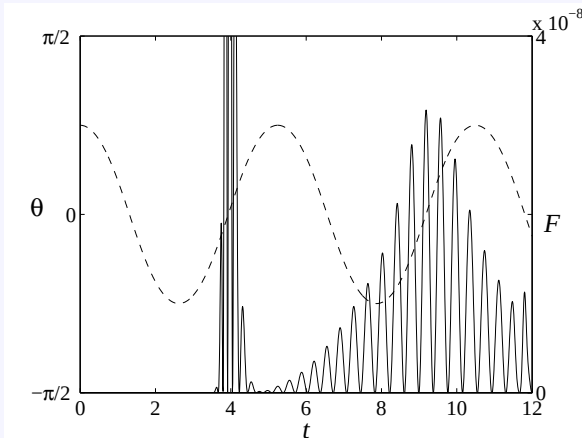
Comparison of asymptotic results with numerical simulations:
initial conditions on unperturbed ($\epsilon = 0$) separatrix.



Plots of $\theta(t)$, $F(t)$ for $s = 25$, $x(s, t) - x_{\text{bal}}^{(0)}(s, \theta(t)) = O(\epsilon^3)$ and $x_w(s, t)$,
for $\epsilon = 0.15$, $a = 2$ and $b = 1$.

Small-Rossby-number regime (continued)

Initial conditions on unperturbed ($\epsilon = 0$) periodic orbit $\theta(0) = \pi/4$, $w(0) = 0$.



Plots of $\theta(t)$, $F(t)$ for $s = 25$, $x(s, t) - x_{\text{bal}}^{(0)}(s, \theta(t)) = O(\epsilon^3)$ and $x_w(s, t)$ for $\epsilon = 0.125$, $a = 2.5$ and $b = 0.5$.

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