The breakdown of balance in geophysical flows

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Balance and the slow manifold

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- $\gg$ 1 day,

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Mid-latitude atmosphere: rapidly rotating, strongly stratified fluid,

- supports inertia-gravity waves (IGWs), with

$$f < \omega < N,$$

  where Coriolis parameter $f \approx 10^{-4}\,\mathrm{s}^{-1}$ and Brünt–Väisälä frequency $N = |g\rho_z/\rho|^{1/2} \approx 10^{-2}\,\mathrm{s}^{-1}$

- possible time scales: minutes $< T_{\mathrm{igw}} <$ hours

Balance and the slow manifold (continued)

Small parameter: Rossby number

$$\epsilon = \frac{T_{\mathrm{igw}}}{T_{\mathrm{a}}} < \frac{U}{fL} = R \ll 1$$

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Dimensionless PDEs: with T_{a} as reference time,

$$\frac{\partial s}{\partial t} = N_s(s, f; \epsilon), \qquad \frac{\partial f}{\partial t} + \mathcal{L} f = N_f(s, f; \epsilon)$$

with $\operatorname{spec} \mathcal{L} = \{\mathrm{i}\omega : \omega \in \mathbf{R}, |\omega| > \epsilon^{-1}\}$.

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- ○ slow (nonlinear) dynamics, s, $T = O(1)$,
- ○ fast inertia-gravity waves, f, $T > O(\epsilon^{-1})$.

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Observations:

- ∘ fast oscillations are weak, $f \ll s$

Balance and slow manifold

Balanced motion: devoid of inertia-gravity waves (no fast oscillations)

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At higher orders: slaving (Warn et al)

$$f = F(s) = \epsilon F_1(s) + \epsilon^2 F_2(s) + \cdots,$$

defines a slow manifold $(\dim s = \dim f\,/2)$.

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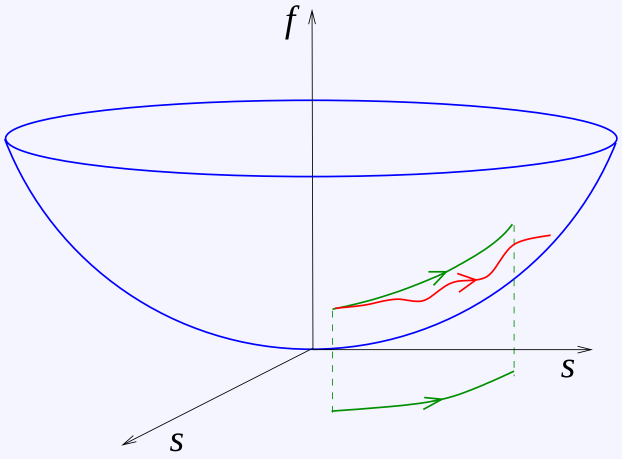

$F(s)$ can be computed to $O(\epsilon^N)$; the corresponding manifold is approximately invariant (invariant at $O(\epsilon^N)$).

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(quasi-, semi-geostrophic, L1, BE...)

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Questions:

- Can balance be perfect?
  Is there an invariant slow manifold?
- If not, describe the spontaneous generation of inertia-gravity waves. What is the wave amplitude?

Balance, slow manifold (continued)

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Here: use exponential asymptotics to estimate IGW amplitude

Lorenz’s 5-component model

Lorenz’s less-famous equations:

Shallow-water on the sphere, spectral expansion, truncation to 3 modes → 9-component model

Neglect 2 IGWs → 5-component model (Lorenz 1986):

$$\begin{aligned}
\dot{u} &= -vw + \epsilon bvy \\
\dot{v} &= uw - \epsilon buy \\
\dot{w} &= -uv \\
\epsilon\dot{x} &= -y \\
\epsilon\dot{y} &= x + buv
\end{aligned}$$

Slow variables: $s = (u, v, w)$ Rossby-wave triad

Fast variables: $f = (x, y)$ inertia-gravity-wave pair.

Lorenz's 5-component model (continued)

2 parameters:

- $b = fL/\sqrt{gH}$ = rotational Froude number = coupling parameter,
- $\epsilon = T_{\mathrm{a}}/T_{\mathrm{igw}} = bR/\sqrt{1+b^2}$ = (wave frequency)$^{-1}$,

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Quasi-geostrophic regime: $b = O(1),\ \ \epsilon = O(R) \ll 1$.
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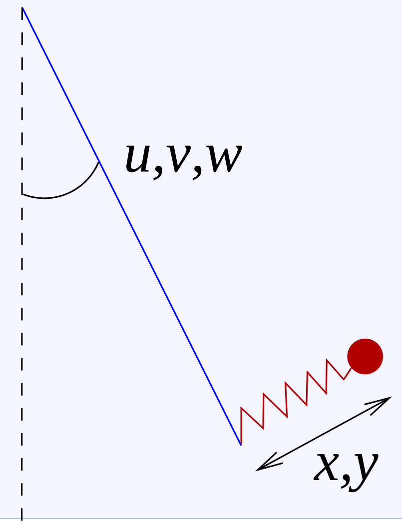
Lorenz’s model: a pendulum coupled to a stiff spring



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Lorenz’s 5-component model (continued)

At leading order in ϵ, balanced (quasi-geostrophic) dynamics

$$\dot{u} = -vw, \quad \dot{v} = uw, \quad \dot{w} = -uv$$

decouples from IGW-dynamics, $\epsilon\dot{x} = y, \quad \epsilon\dot{y} = x$.

Balanced motion, free of IGWs at $O(\epsilon)$: $x, y = 0 + O(\epsilon)$, geostrophic balance.

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Higher-order balance: slaving

$$x = \hat{x}(u,v,w) = \sum_{n=1}^{N} \epsilon^n \hat{x}_n(u,v,w), \quad y = \hat{y}(u,v,w) = \sum_{n=1}^{N} \epsilon^n \hat{y}_n(u,v,w)$$

The $(\hat{x}_n, \hat{y}_n)$ are polynomials:

$$\hat{x}_n = (2n)! \sum_{i,j=0}^{n} C_{ij}^{n}\, u^{2i+1} v^{2j+1} w^{2(n-i-j)}$$

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Exponential asymptotics

The coefficients C_{ij}^n are easily computed by recurrence.
Remarkably, they have simple asymptotics as $n \to \infty$:

$$C_{ij}^n \sim (-1)^{j+1} f\left(\frac{i-n/3}{\sqrt{n}}, \frac{j-n/3}{\sqrt{n}}\right), \quad f(\xi,\eta) = \mathrm{e}^{-15(\xi^2+\xi\eta+\eta^2)/2]}$$

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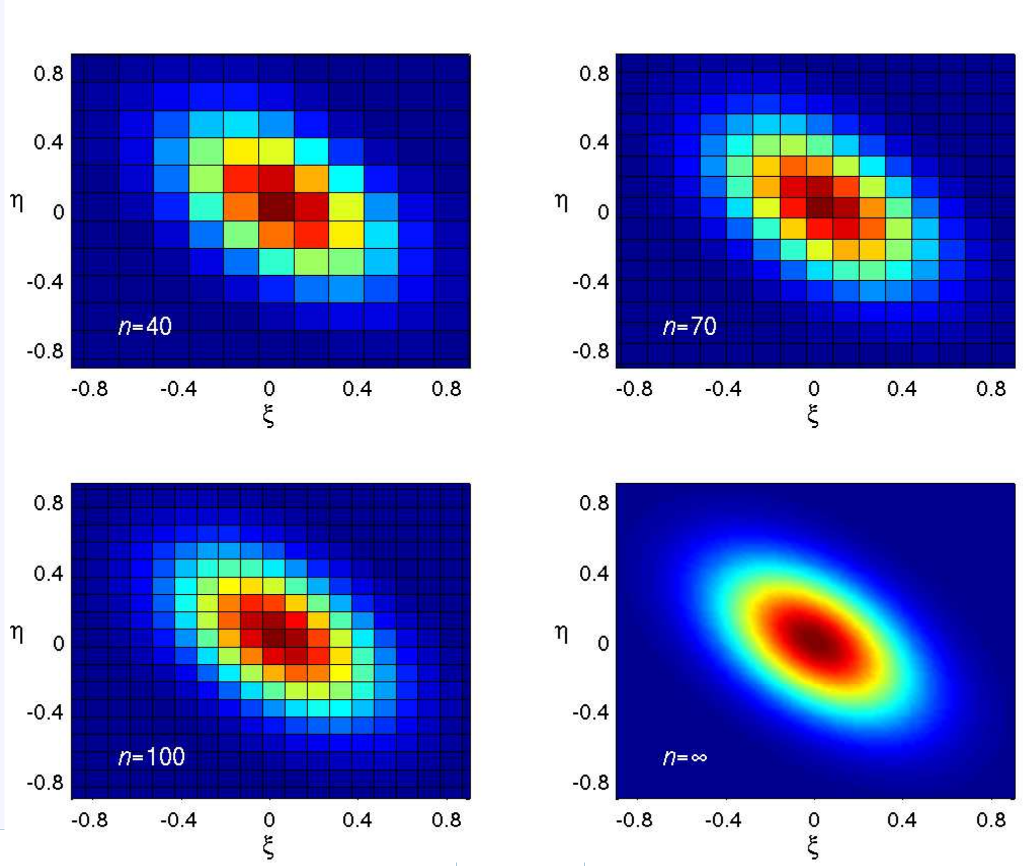

Exponential asymptotics

For $\epsilon = 0$, consider the balanced solution

$$u_0 = \operatorname{sech} t, \;\; v_0 = -\tanh t, \;\; w_0 = -\operatorname{sech} t,$$

$$x_0 = -b u_0 v_0, \;\; y_0 = 0.$$

For $\epsilon > 0$, regular perturbation expansion $u_{\text{bal}} = \sum_{n=0}^{N} \epsilon^n u_n$.
Balanced solution, slow, symmetric: $y_{\text{bal}}(-t) = y_{\text{bal}}(t), \quad \cdots$

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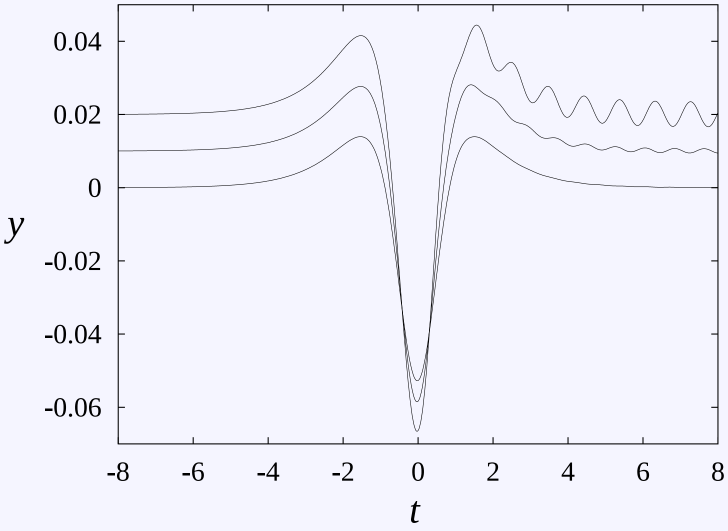
Numerical solution:



IGW generation is obvious.

Exponential asymptotics (continued)

Use exponential asymptotics to estimate IGW amplitude.

IGW generation corresponds to a Stokes phenomenon, associated with poles of $u_{\mathrm{bal}}(t)$, $v_{\mathrm{bal}}(t)$ at $t = \pm \mathrm{i}\pi/2$: homogeneous solution

$$x_{\mathrm{igw}}(t) = C\cos(t/\epsilon + \phi) \quad y_{\mathrm{igw}}(t) = C\sin(t/\epsilon + \phi)$$

is switched on across the Stokes line $\mathrm{Re}\, t = 0$

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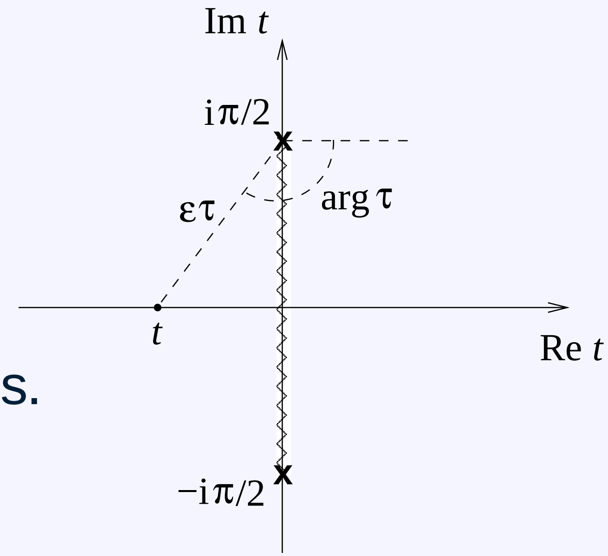

- for $t < 0$, $C = 0$: balanced motion,
- For $t > 0$, $C \neq 0$: balanced motion + IGWs.

Exponential asymptotics (continued)

Estimate C using matched asymptotics (Kruskal–Segur, Hakim): near $\pm\mathrm{i}\pi/2$,

$$x_{\mathrm{igw}}(t) \sim C\exp(\pm\mathrm{i}t/\epsilon) = O(1).$$

Thus, $C \propto \exp[-\pi/(2\epsilon)]$.

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To estimate prefactors, rescale the Lorenz equations near $t = \pm \mathrm{i}\pi/2$, with $t = \mathrm{i}\pi/2 + \epsilon\tau$, and

$$u(t) = \epsilon^{-1} U(\tau), \quad x(t) = \epsilon^{-2} X(\tau), \cdots .$$

This gives the inner equations

$$U' = -VW + bVY, \quad V' = UW - bUY, \quad W' = -UV,$$
$$X' = -Y, \quad Y' = X + bUV.$$

Lorenz's equations with $\epsilon = 1$!

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Exponential asymptotics (continued)

Need solution for $|\tau| \to \infty$ for matching: expand in inverse powers of τ, e.g.

$$Y(\tau) = \sum_{n=1}^{\infty} \frac{Y_n}{\tau^{2n+1}}$$

Divergent series: $Y_n \sim \mathrm{i}(-1)^n (2n)!\,\kappa$ for some $\kappa = \kappa(b)$.

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Use Borel summation:

$$Y = \frac{1}{\tau} \int_0^{\infty} \mathrm{e}^{-s}\,\mathrm{d}s \sum_{n=1}^{\infty} \frac{Y_n}{(2n)!} \left(\frac{s}{\tau}\right)^{2n}$$

with integration contour chosen to match $y = y_{\mathrm{bal}} + C\sin(t/\epsilon + \phi)$ with $C = 0$ for $t < 0$. Analytic continuation gives

$$C = \frac{2\pi\kappa}{\epsilon^2} \exp\left(-\frac{\pi}{2\epsilon}\right) \quad \text{for } t > 0.$$

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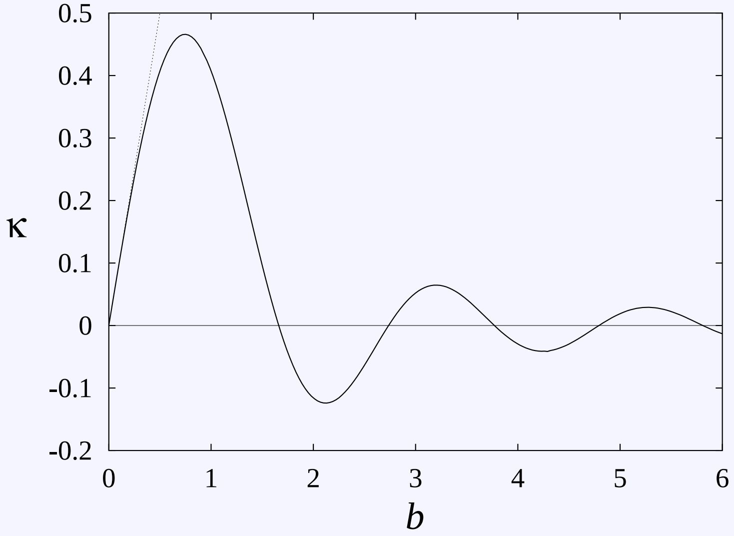
Exponential asymptotics

κ as a function of the rotational Froude number b is found from recurrence relation:

$$\kappa = \lim_{n\to\infty} \frac{(-1)^n Y_n}{\mathrm{i}(2n)!}$$



For $b \ll 1$, $\kappa \sim b$ (Lorenz–Krishnamurthy)

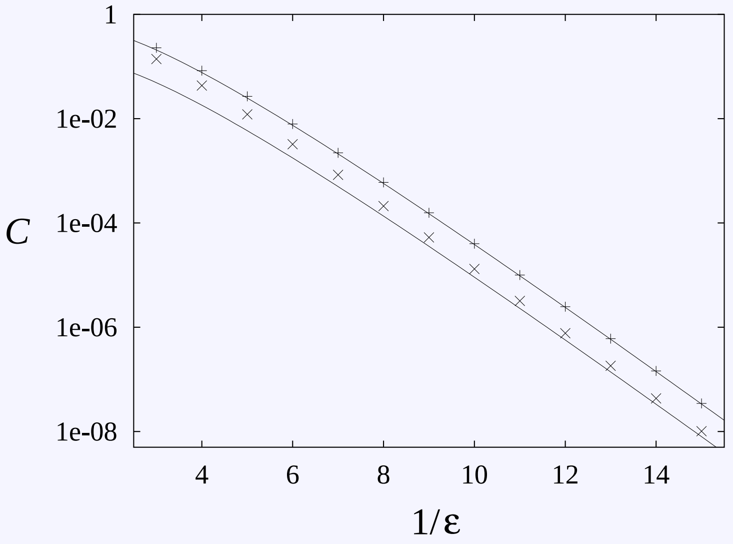
Comparison with numerics

Estimate numerically $(x_{\rm igw}, y_{\rm igw})$: solve numerically the Lorenz equations for (u, v, w, x, y), estimate the balanced component using slaving,

$$x_{\rm bal}(t) \approx \sum_{n=1}^{N} \epsilon^n \hat{x}_n(u(t), v(t), w(t)),$$

choose N for optimal truncation.

Then, $x_{\rm igw}(t) \approx x(t) - x_{\rm bal}(t)$ and $C = \left[x_{\rm igw}^2 + y_{\rm igw}^2\right]^{1/2}$ for $t > 0$.



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Comparison with numerics (continued)

Remarks:

- IGWs generated smoothly near $t = 0$ (Berry 89):

$$(x_{\mathrm{igw}}^2 + y_{\mathrm{igw}}^2)^{1/2} \approx \frac{C}{2}\left[1 + \mathrm{erf}\left(\frac{t}{(\pi\epsilon)^{1/2}}\right)\right]$$

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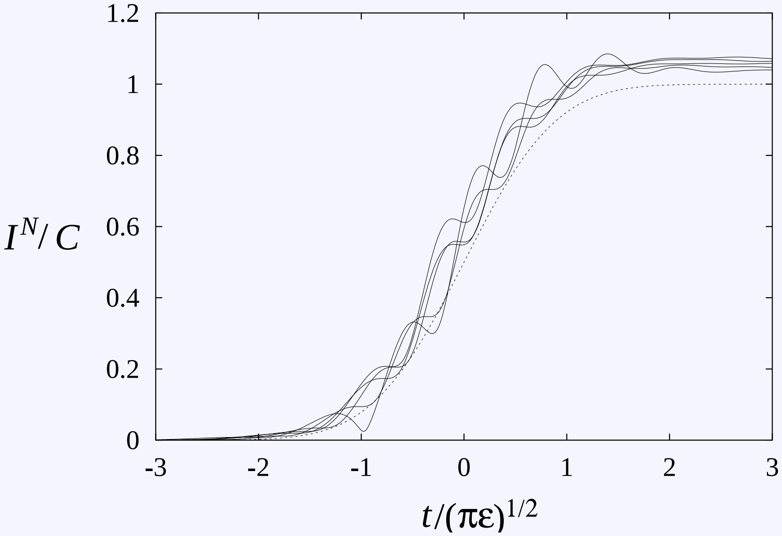

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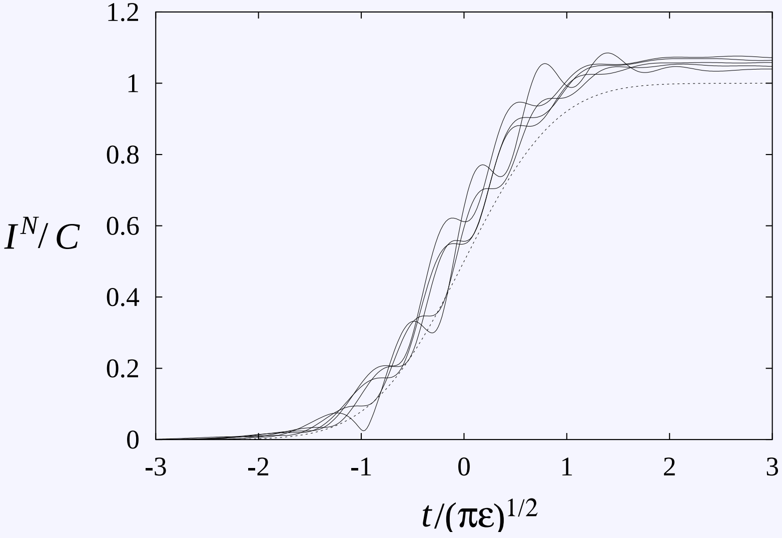
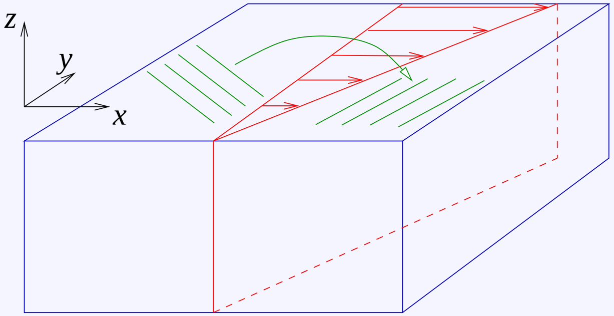


- Generation from periodic solutions: $u_0 = \mathrm{cn}\,(t/k, k),\ \ldots,$

$$C = \frac{2\pi\kappa}{\epsilon^2}\mathrm{e}^{-kK(\sqrt{1-k^2})/\epsilon}$$

Sheared-disturbance model

Inertia-gravity-wave generation for solutions of 3D fluid equations (with I Yavneh)



In 3D Boussinesq PDE, consider solution

$$u = \underbrace{\Sigma y}_{\text{shear flow}} + \nu(t)\underbrace{\exp[\mathrm{i}(kx - kt)y + mz]}_{\text{plane wave}}$$

$\longrightarrow$ exact reduction to an ODE (Kelvin)

$$\epsilon^2 \left[\ddot{\zeta} + b(t)\dot{\zeta}\right] + c(t)\zeta = d(t),$$

for amplitude of vorticity $\zeta(t)$.

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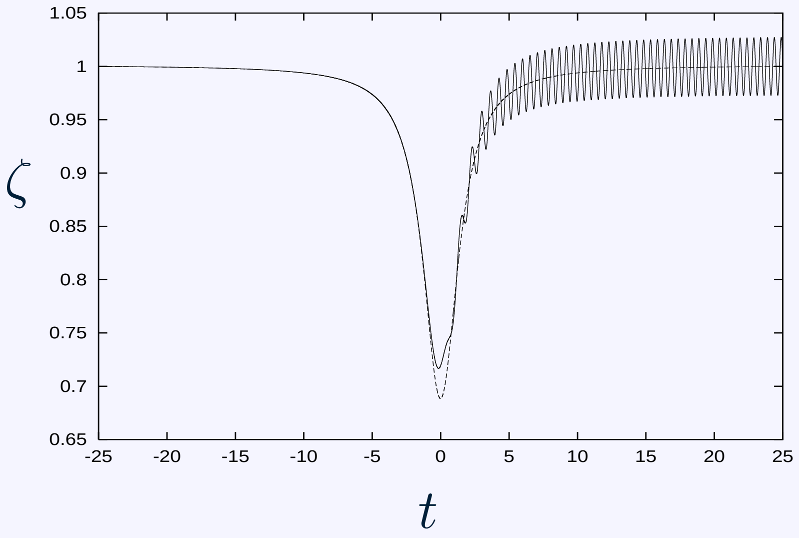
Sheared-disturbance model (continued)

Parameters: Rossby number $\epsilon = |\Sigma|/f$, aspect ratio $\delta = m/k$, and Prandtl ratio $S = N^2/f^2 > 1$.

For $|t| \to \infty$,

$$\zeta \sim \underbrace{1}_{\text{balanced}} + \underbrace{C\cos(S^{1/2}t/\epsilon + \phi)}_{\text{inertia}-\text{gravity waves}}.$$

- For $t \to -\infty$, assume balance $C = 0$.
- For $t \to \infty$, find $C \propto \exp(-\alpha/\epsilon) \neq 0$: IGWs generation.

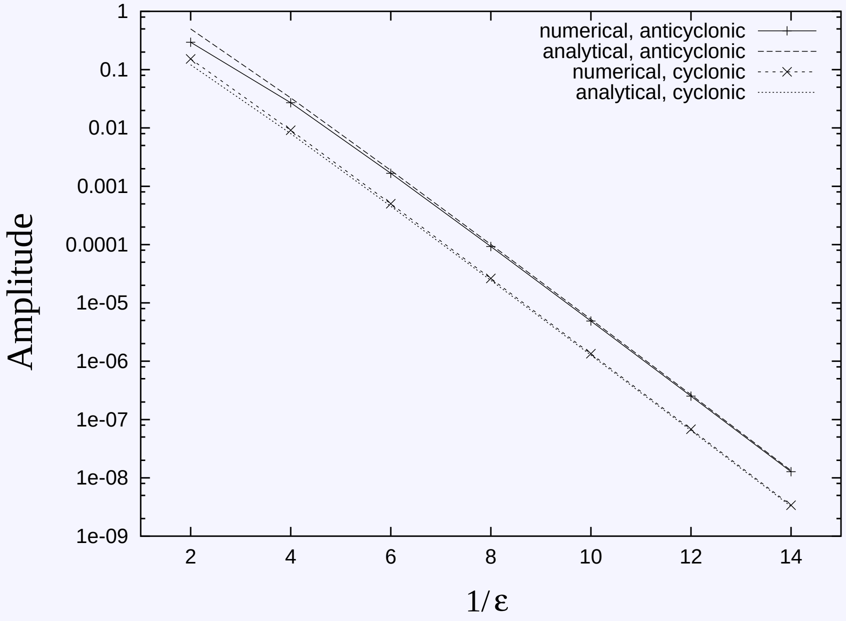


Sheared-disturbance model (continued)

Exponential asymptotics provides an estimate for C:

$$C \sim \frac{(2\pi)^{1/2}\delta\, \mathrm{e}^{\pm\beta}}{(S+\delta^2)^{1/4}}\ \frac{\mathrm{e}^{-\alpha/\epsilon}}{\epsilon^{1/2}}$$

where $\alpha = \alpha(S,\delta)$ and $\beta(S,\delta)$ are elliptic integrals.



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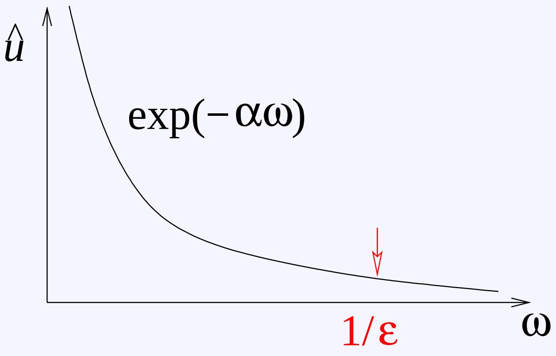

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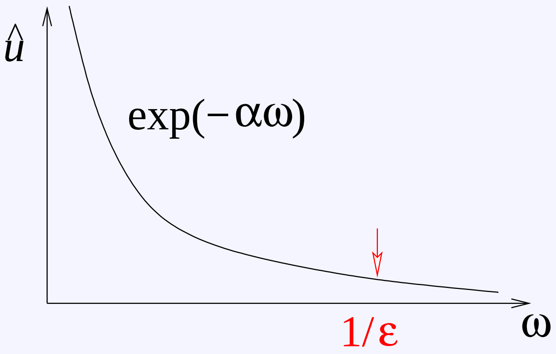

- Importance of time-scale separation: vs. Lighthill radiation for $b \ll 1$, $\epsilon = O(1)$ (Ford et al . . .).

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Open problems

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- Dissipation mechanisms: slow motion inefficient (inverse cascade) but very slow transfer to (forward cascading) fast motion (Warn, Straub, McWilliams).