

Scalar decay in smooth random flows

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with P H Haynes (Cambridge)

Scalar decay

Concentration $C(\mathbf{x}, t)$ of a passive scalar released in a flow obeys

$$\partial_t C + \mathbf{v} \cdot \nabla C = \kappa \Delta C, \quad \nabla \cdot \mathbf{v} = 0.$$

Focus on **spatially smooth flows**: $|\mathbf{v}(\mathbf{x}, t) - \mathbf{v}(\mathbf{0}, t)| \sim |\mathbf{x}|$ for small $|\mathbf{x}|$ at the scale of concentration fluctuations.

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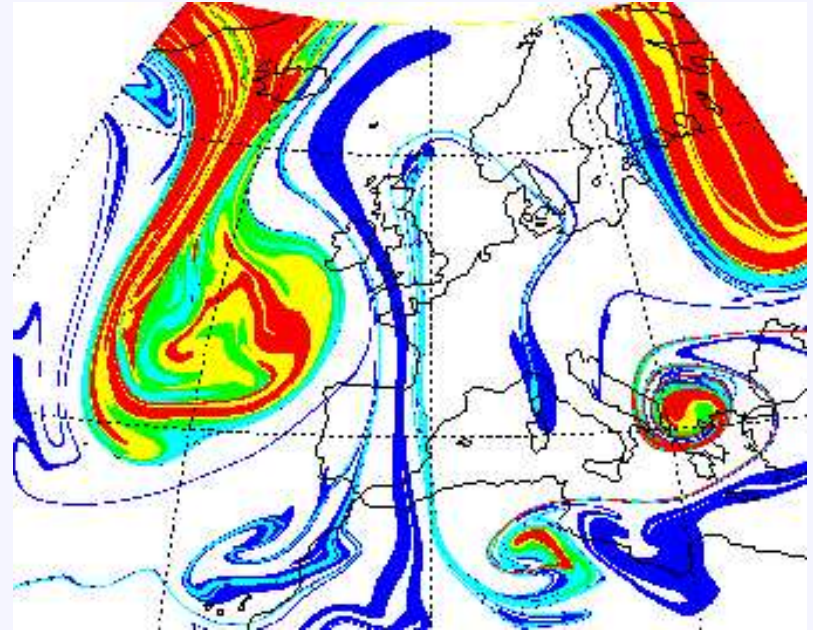
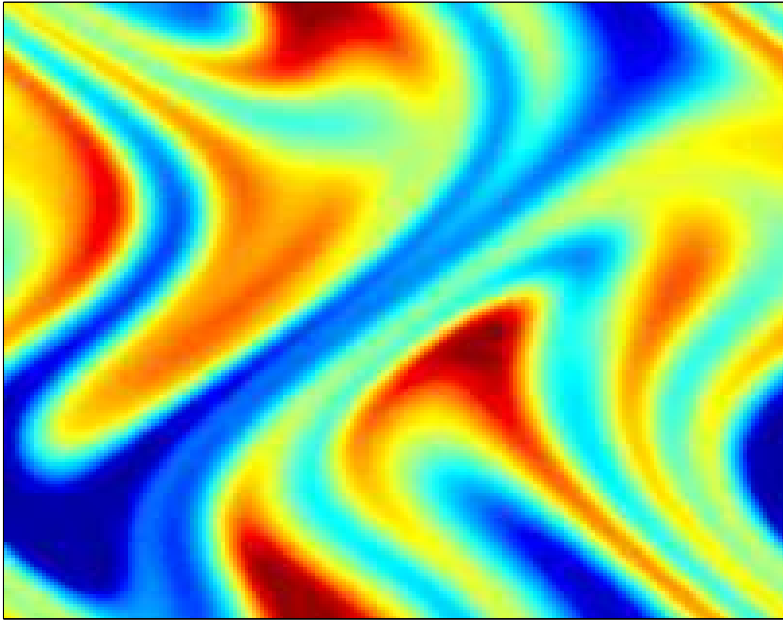
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Large-scale-dominated flows:

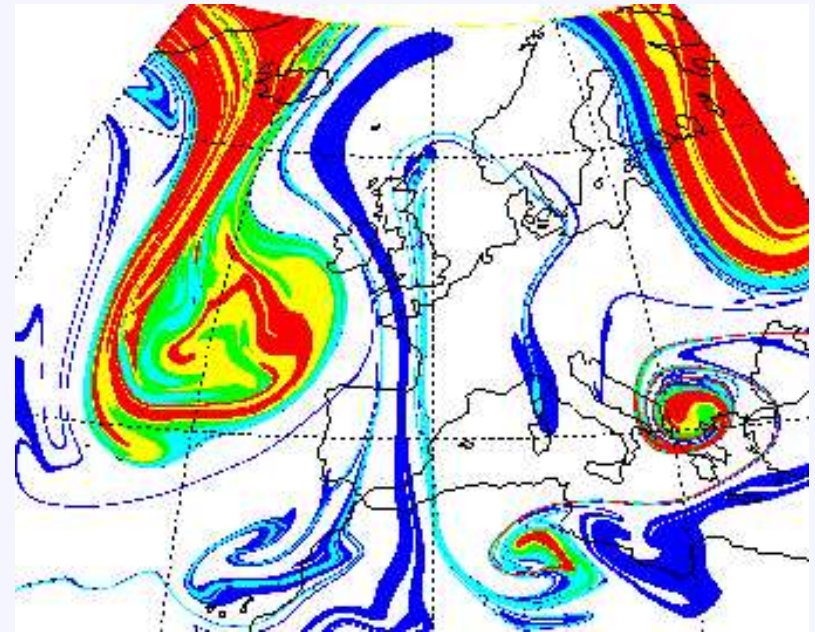
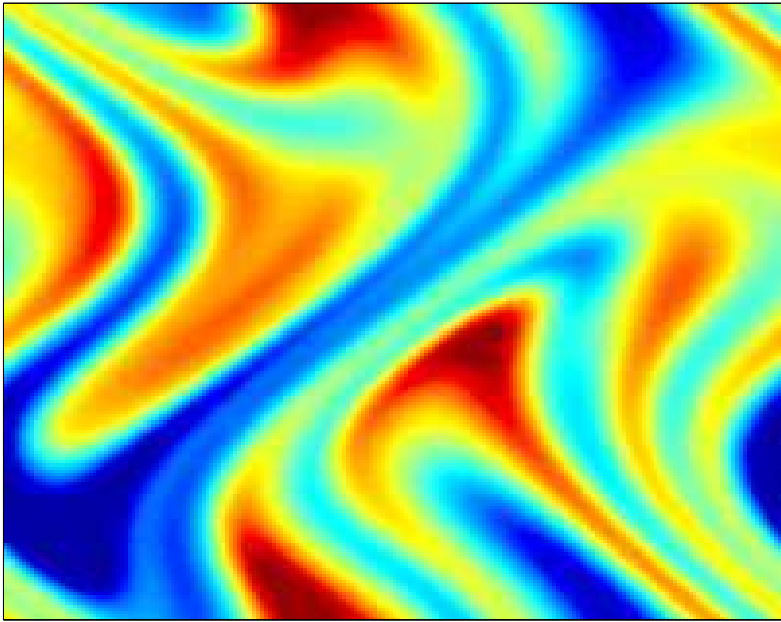
- Batchelor regime: $\nu \gg \kappa$;
- Steep energy spectrum: $E(k) = o(k^{-3})$ as $k \rightarrow \infty$;
- 2D turbulence (enstrophy cascade);
- Large-scale geophysical flows.

But not 3D turbulence ($E(k) \sim k^{-5/3}$).

Scalar decay



Scalar decay



Assume $\bar{C} = 0$; in bounded domains, C decays exponentially as $t \rightarrow \infty$:

$$C \sim \exp(-\gamma_C t), \quad \gamma_C > 0.$$

What controls γ_C , in particular for $\kappa \rightarrow 0$?

Scalar decay

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For mixing flows, $\gamma_C \rightarrow \gamma_C^0 \neq 0$ as $\kappa \rightarrow 0$.

What controls γ_C^0 ? What is the diffusive correction for $\kappa \neq 0$?

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Here: we show that either case is possible, depending on flow and domain size.

Focus on variance decay rate γ_2 ,

$$\langle \overline{C^2} \rangle \sim \exp(-\gamma_2 t)$$

(In numerical simulations, it appears that $\gamma_2 \approx 2\gamma_C$.)

Lagrangian stretching

The distance $\mathbf{d}$ between nearby trajectories in $\mathbf{v}$ evolves according to

$$\dot{\mathbf{d}} = \mathbf{d} \cdot \nabla \mathbf{v},$$

and grows exponentially for $t \rightarrow \infty$.

The growth is characterized by finite-time stretching factors

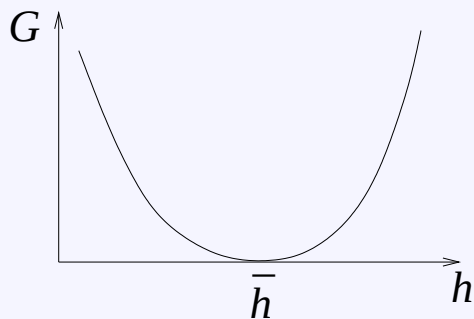
$$h := \frac{1}{t} \log \frac{d(t)}{d(0)}, \quad \text{where } d = |\mathbf{d}|.$$

The pdf of h has the large-deviation approximation,

$$p(h; t) \propto \exp[-tG(h)],$$

for some Cramer function $G(h)$.

Lagrangian stretching



(Stretching) Liapunov exponent $\bar{h}$ satisfies $G(\bar{h}) = G'(\bar{h}) = 0$.

Moments of d evolve according to

$$\langle d^m \rangle = \int_0^\infty e^{mht} p(h; t) dh \propto \exp[F(m)t]$$

where the free energy

$$F(m) = \sup_h [mh - G(h)].$$

(Intermittency: $F(m) \neq m\bar{h}$.)

Lagrangian stretching

Regard C as a superposition of simple functions, experiencing different stretching histories with pdf $p(h; t)$, then

$$\langle C^2 \rangle \propto \int_0^\infty e^{-ht} p(h; t) dh \propto \int_0^\infty e^{-t[h+G(h)]} dh.$$

(Antonsen et al, Balkovsky & Fouxon).

Approximating the integral gives the decay rate:

$$\gamma_2^L = \begin{cases} -F(-1) & \text{if } F'(-1) \geq 0 \\ G(0) & \text{if } F'(-1) \leq 0 \end{cases}$$

Attempts to confirm this result numerically have given ambiguous results.

Covariance equation

Consider classes of flows for which the covariance

$$\Gamma(\mathbf{x}, t) := \langle C(\mathbf{x} + \mathbf{y}, t) C(\mathbf{y}, t) \rangle$$

satisfies a closed equation.

- **Kraichnan flows**: short correlation time, white noise;
- **renewing flows**: complete decorrelation in finite time, independent random maps.

The results are identical in both cases. Describe the Kraichnan case:

$$\langle v_i(\mathbf{x}, t) v_j(\mathbf{x}', t') \rangle = 2B_{ij}(\mathbf{x} - \mathbf{x}') \delta(t - t')$$

Decay in Kraichnan flows

The covariance satisfies the linear equation

$$\partial_t \Gamma = 2\kappa \Delta \Gamma + 2D_{ij}(\mathbf{x}) \partial_{ij}^2 \Gamma, \quad D_{ij}(\mathbf{x}) := B_{ij}(0) - B_{ij}(\mathbf{x}).$$

The variance decay rate γ_2 is the smallest eigenvalue λ of

$$2 [\kappa \Delta + D_{ij}(\mathbf{x}) \partial_{ij}^2] \Gamma = -\lambda \Gamma$$

Study the spectrum of a linear operator.

Boundary conditions: doubly periodic domain $[0, 2P\pi] \times [0, 2P\pi]$.

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$$2 \left[\kappa \Delta + D_{ij}(\mathbf{x}) \partial_{ij}^2 \right] \Gamma = -\lambda \Gamma$$

For $\kappa = 0$, the operator is singular at $\mathbf{x} = 0$:

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Two different regimes of variance decay:

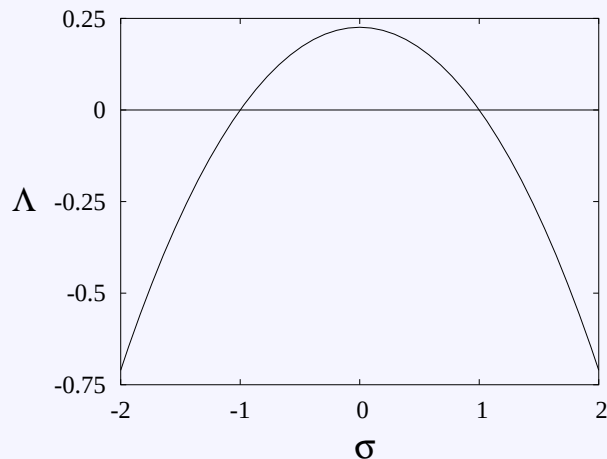
- Local control: in the absence of discrete eigenvalue, $\gamma_2 \approx \ell$;
- Global control: in the presence of discrete eigenvalues $\lambda < \ell$,
 $\gamma_2 \approx \lambda$.

Local eigenvalue problem

For $\kappa = 0$ and $\mathbf{x} \rightarrow 0$, the eigenvalue problem reduces to

$$S_{ijkl}x_kx_l\partial_{ij}^2\Gamma = \lambda\Gamma, \quad S_{ijkl} := -\partial_{kl}^2D_{ij}(\mathbf{x})|_{\mathbf{x}=0},$$

and admits simple solutions $\Gamma(\mathbf{x}) = r^{\sigma-1}f_{\sigma}(\theta)$.



This defines a real function $\Lambda(\sigma) = \lambda$.

For $\lambda > \Lambda_{\max}$, solution oscillates as $\mathbf{x} \rightarrow 0$:
the limit of the continuous spectrum is
 $\ell = \Lambda_{\max}$.

In 2D, $\Lambda(\sigma) = \Lambda(-\sigma)$, hence $\ell = \Lambda(0)$.

Matched asymptotics

For $\kappa \neq 0$, matched asymptotics can be used to find approximations to the eigenvalues.

Near ℓ , there is an eigenvalue

$$\lambda = \ell - \frac{2\pi^2 \Lambda''(0)}{\log^2 \kappa} + o(1/\log^2 \kappa).$$

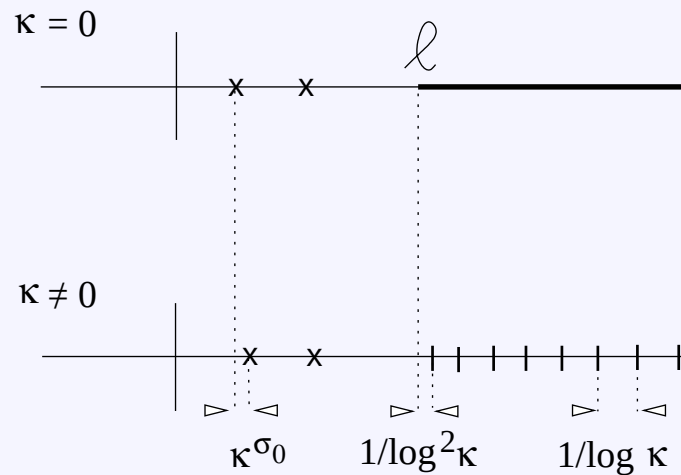
Other eigenvalues emerging from the continuous spectrum are separated by $O(1/\log \kappa)$ gaps.

Near discrete eigenvalues λ_0 , the eigenvalues for $\kappa = 0$ are

$$\lambda = \lambda_0 + c\kappa^{\sigma_0} + o(\kappa^{\sigma_0}),$$

where $\lambda_0 = \Lambda(\sigma_0)$.

Spectrum



- **Local control:** no isolated discrete eigenvalues, $\gamma_2 \approx \ell$;
- **Global control:** isolated eigenvalues, $\gamma_2 \approx \lambda_0 < \ell$.

Expect local control when domain size $\approx$ flow size, global control when domain size $\gg$ flow size.

Connection with Lagrangian stretching theories

$$\gamma_2^L = \begin{cases} -F(-1) & \text{if } F'(-1) \geq 0 \\ G(0) & \text{if } F'(-1) \leq 0 \end{cases}$$

The Fokker–Planck equation for $\mathbf{d}$ is identical to the local eigenvalue problem for Γ : as a result

$$F(m) = -\Lambda(-m - 1).$$

It follows that

$$\ell = G(0).$$

In 2D, $\Lambda(\sigma) = \Lambda(-\sigma)$ implies $F(m) = F(-m - 2)$ and

$$\ell = G(0) = -F(-1).$$

Lagrangian stretching theories correctly predict γ_2 in the locally controlled case.

Example

We take $\mathbf{v} = (-\psi_x, \psi_y)$, with

$$\psi = \cos(x + \phi_1(t)) + \cos(y + \phi_2(t)),$$

and ϕ_i independent, short-correlation phases.

The corresponding eigenvalue problem

$$-2\kappa\Delta\Gamma - 4 \left[\sin^2(y/2)\partial_{xx}^2 + \sin^2(x/2)\partial_{yy}^2 \right] \Gamma = \lambda\Gamma$$

can be solved numerically for small κ for different domain sizes P .

The local version

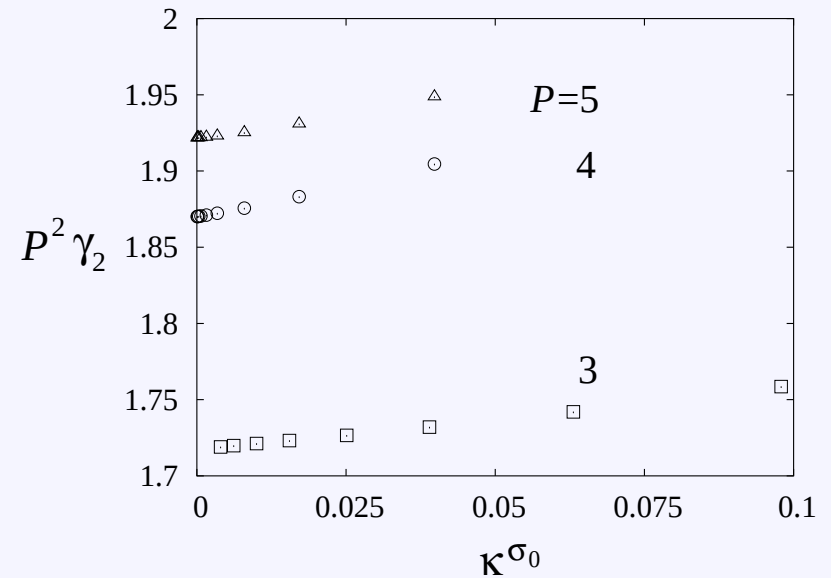
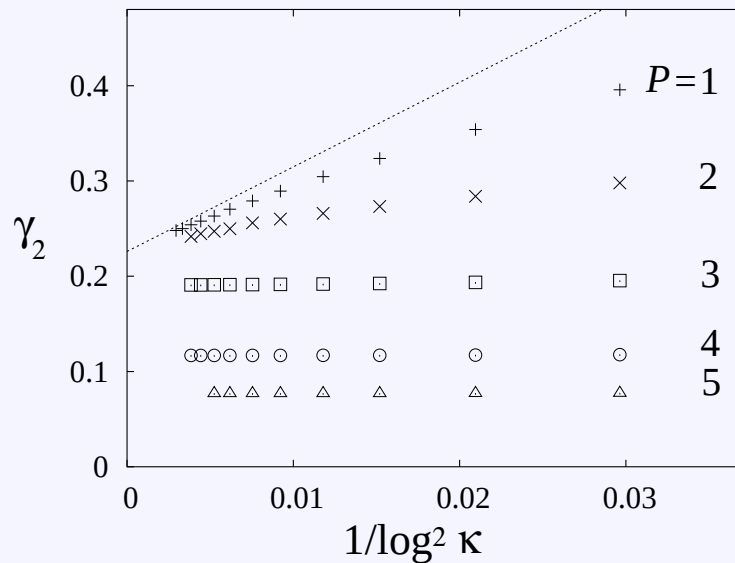
$$- (y^2\partial_{xx}^2 + x^2\partial_{yy}^2) \Gamma = \lambda\Gamma$$

and solutions $r^{\sigma-1}f_\sigma(\theta)$ gives $\Lambda(\sigma)$, with $\ell = \Lambda(0) = 0.226$.

A Rayleigh quotient argument indicates that there are isolated discrete eigenvalues $\lambda_0 < 2/P^2$: **global control** for $P \geq 3$.

Example

Decay rate γ_2 as a function of κ :



- Local scaling: $\gamma_2 \approx 0.226 + 0.88/\log^2 \kappa$;
- Global scaling: $\gamma_2 \approx \lambda_0 + c\kappa^{\sigma_0}$.
- Homogenization: $\gamma_2 \approx 2/P^2 + c\kappa$.

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- The different control mechanisms can be identified with different parts of spectrum of 'transfer operator' for $\kappa = 0$.
- Results obtained for Kraichnan flows and renewing flows: presumably also hold for flows with finite correlation time.
- In the locally controlled case, the convergence to zero diffusivity limit is slow ($1/\log^2 \kappa$).

Open issues

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How does γ_n depend on n ? In numerical experiments

$$\gamma_n \approx n\gamma_C.$$

Equivalently, what is the distribution of **finite-time decay rates**

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To start with, consider variance $\langle (\gamma_t - \langle \gamma \rangle)^2 \rangle$:

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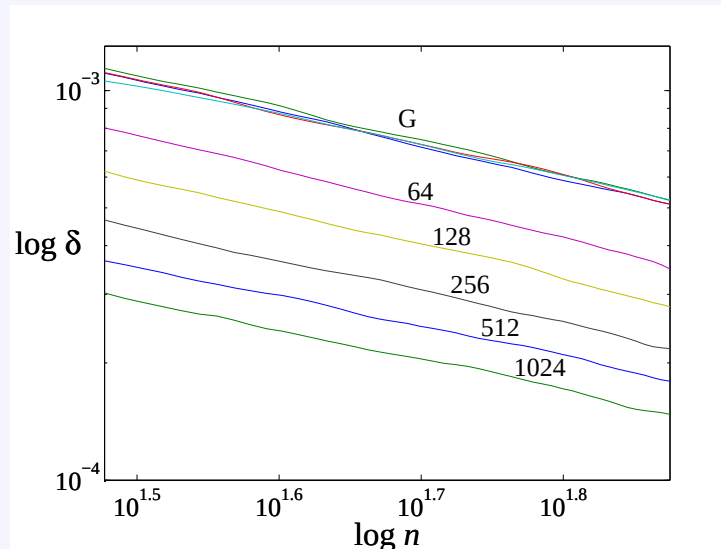
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Open issues

- Conditions on $\mathbf{v}$ for $\gamma_C \rightarrow \gamma_0 \neq 0$ as $\kappa \rightarrow 0$?
- Relationship to ergodic theory for $\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}, t) + \sqrt{2\kappa}\dot{\mathbf{W}}$.
- Scalar decay in flows with transport barriers (invariant curves):

$$\gamma_C \rightarrow 0 \quad \text{as} \quad \kappa \rightarrow 0.$$

Scaling?