Wave interactions

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Linear waves in conservative systems

Small amplitude motion: linearize equations of motion and introduce solutions proportional to

$$\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)]$$

for wavevector $\boldsymbol{k}$ and frequency ω, related by a dispersion relation, $\omega=\omega(\boldsymbol{k})$.
Example: Rossby waves, large-scale atmospheric and oceanic waves, described by 2D fluid on a β-plane:

$$\zeta_t+\beta\psi_x+\psi_x\zeta_y-\psi_y\zeta_x=0,$$

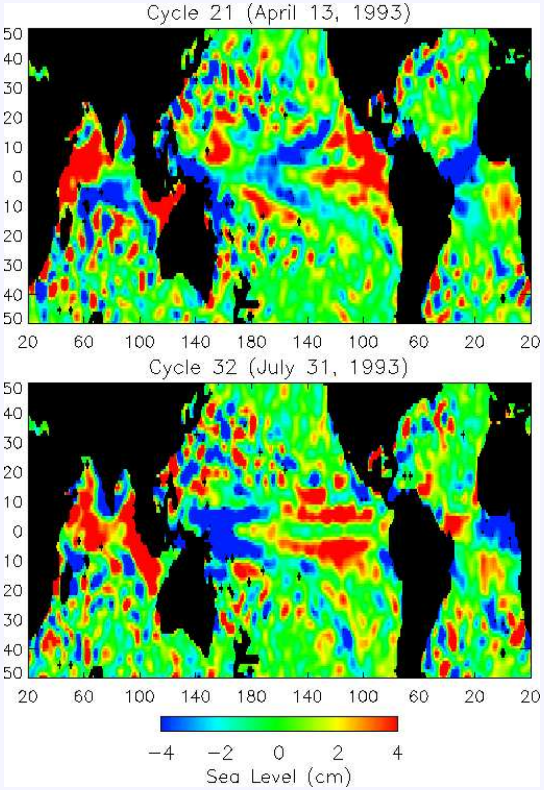
with streamfunction ψ and vorticity $\zeta=\nabla^2\psi$.

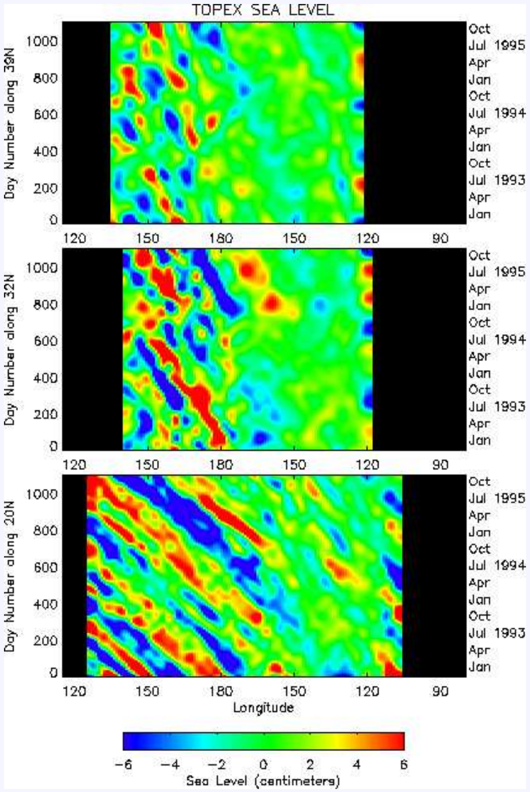
Linear waves: Rossby waves

With $\zeta = A\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)] + \text{c.c.}$, the dispersion relation is found as

$$\omega = -\frac{\beta k}{k^2 + l^2}$$

Note $\omega/k < 0$: eastward propagation.





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Linear waves: Rossby waves

General solution: superposition

$$\zeta = \sum_a A_a \exp[\mathrm{i}(\boldsymbol{k}_a \cdot \boldsymbol{x} - \omega_a t)],$$

with $\sum_a = \sum$ or $\int$, and $A_a^* = A_{-a}$.

The nonlinear equation of motion conserves

energy $\mathcal{E} = \frac{1}{2}\int |\nabla\psi|^2 \,\mathrm{d}\boldsymbol{x}$

pseudomomentum $\mathcal{P} = -\frac{1}{2\beta}\int \zeta^2 \,\mathrm{d}\boldsymbol{x}$

In terms of A_a, these are

$$\mathcal{E} = \frac{1}{2}\sum_a E_a |A_a|^2, \qquad \mathcal{P} = \frac{1}{2}\sum_a P_a |A_a|^2,$$

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Linear waves: general formulation

Consider the system governed by

$$\boldsymbol{u}_t = \mathsf{L}\boldsymbol{u} + \boldsymbol{N}(\boldsymbol{u}),$$

where $\boldsymbol{u} \in \mathbb{R}^n$, L matrix operator, and $N(\boldsymbol{u})$ contains nonlinear terms. Assume two conserved quantities: pseudoenergy and pseudomomentum

$$\mathcal{E} = \mathcal{E}^{(2)} + \mathrm{h.o.t.} \quad \text{and} \quad \mathcal{P} = \mathcal{P}^{(2)} + \mathrm{h.o.t.},$$

quadratic at leading order, with

$$\mathcal{E}^{(2)} = \frac{1}{2}\int \boldsymbol{u}^\dagger \mathsf{E}\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} \quad \text{and} \quad \mathcal{P}^{(2)} = \frac{1}{2}\int \boldsymbol{u}^\dagger \mathsf{P}\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}.$$

(L, E and P are related: $\mathsf{L} = \mathsf{JE}$ for J skew-adjoint , and P is then such that $-\boldsymbol{u}_x = \mathsf{JP}\boldsymbol{u}$.)

Linear waves: general formulation

Introducing wave solutions $\boldsymbol{u} = A\hat{\boldsymbol{u}} \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)] + \mathrm{c.c.}$ leads to the eigenvalue problem

$$-\mathrm{i}\omega\hat{\boldsymbol{u}} = \hat{\mathsf{L}}\hat{\boldsymbol{u}}.$$

This gives a dispersion relation $\omega = \omega_p(\boldsymbol{k})$ with n branches $p = 1, 2, \cdots, n$, and polarization relations $\hat{\boldsymbol{u}} = \hat{\boldsymbol{u}}_p$.

The general solution is again obtained by superposition

$$\boldsymbol{u} = \sum_a A_a \hat{\boldsymbol{u}}_a \exp[\mathrm{i}(\boldsymbol{k}_a \cdot \boldsymbol{x} - \omega_a t)],$$

with a denoting wavenumber $\boldsymbol{k}_a$, and branch p_a.

If $(\omega_a, \hat{\boldsymbol{u}}_a)$ is a solution for the wavevector $\boldsymbol{k}_a$, $(\omega_{-a}, \hat{\boldsymbol{u}}_{-a}) = (-\omega_a, \hat{\boldsymbol{u}}_a^*)$ is a solution for the wavevector $-\boldsymbol{k}_a$. Take $A_{-a} = A_a^*$.

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Linear waves: general formulation

The conservation laws imply orthogonality relations: for each $\boldsymbol{k}$,

$$\mathcal{E}^{(2)} = \frac{(2\pi)^d}{2} \sum_{p,q=1}^{n} A_p^* A_q \boldsymbol{u}_p^\dagger \mathsf{E} \boldsymbol{u}_q \exp[\mathrm{i}(\omega_p - \omega_q)t],$$

is constant only if

$$(2\pi)^d \boldsymbol{u}_p^\dagger \mathsf{E} \boldsymbol{u}_q = E_p \delta_{p,q}.$$

with E_p pseudoenergy of branch p: orthogonality in the sense of pseudoenergy.
(Similarly, orthogonality in the sense of pseudomomentum.)

Pseudoenergy and pseudomomentum are related:

$$\frac{E_a}{\omega} = \frac{P_a}{k_a} = \text{wave action}$$

Linear waves: general formulation

Exercise: derive dispersion relation, polarization relation, and show orthogonality for internal gravity waves.

With $\boldsymbol{u} = (\zeta, b)$, they are governed by a system of the general form with

$$\mathsf{L} = \begin{pmatrix} 0 & \partial_x \\ -N^2 \partial_x \nabla^{-2} & 0 \end{pmatrix}, \quad \mathsf{E} = \begin{pmatrix} -\nabla^{-2} & 0 \\ 0 & N^{-2} \end{pmatrix},$$

$$\mathsf{P} = - \begin{pmatrix} 0 & N^{-2} \\ N^{-2} & 0 \end{pmatrix},$$

(Solution: $\omega_p = (-1)^p N k / (k^2 + l^2)^{1/2}$, etc.)

Interaction equations

Weak nonlinearity leads to modulation of wave amplitudes A_a.

Rossby waves: introduce

$$\zeta = \sum_a A_a(t) \exp[\mathrm{i}(\boldsymbol{k}_a \cdot \boldsymbol{x} - \omega_a t)]$$

into the nonlinear equation of motion lead to

$$\dot{A}_a = \frac{1}{2} \sum_{bc} I_a^{bc} A_b^* A_c^* \exp(2\mathrm{i}\Omega_{abc} t) \delta_{\boldsymbol{k}_a + \boldsymbol{k}_b + \boldsymbol{k}_c},$$

where $2\Omega_{abc} = \omega_a + \omega_b + \omega_c$,

The strength of the interaction depends on the interaction coefficient

$$I_a^{bc} = (k_b l_c - k_c l_b) \left[(k_b^2 + l_b^2)^{-1} - (k_c^2 + l_c^2)^{-1} \right]$$

(satisfying $I_a^{bc} = I_a^{cb}$).

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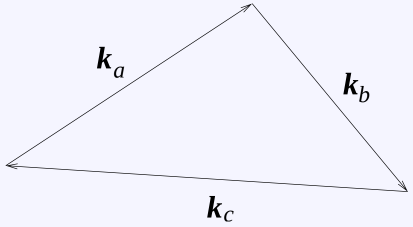
Interaction equations

Interactions are limited to wave triads satisfying the interaction condition



$$\boldsymbol{k}_a + \boldsymbol{k}_b + \boldsymbol{k}_c = 0$$

This appears through the factor

$$\delta_{\boldsymbol{k}_a+\boldsymbol{k}_b+\boldsymbol{k}_c} = \begin{cases} \delta_{k_a+k_b+k_c,0}\,\delta_{l_a+l_b+l_c,0} & \text{periodic b.c.} \\ \delta(k_a+k_b+k_c)\delta(l_a+l_b+l_c) & \text{unbounded domains} \end{cases}$$

General formulation: interaction equations are derived using orthogonality relations, leading to

$$I_a^{bc} = (2\pi)^d \hat{\boldsymbol{u}}_a^\dagger \mathsf{E} \left[\hat{\boldsymbol{N}}^{(2)}(\hat{\boldsymbol{u}}_b, \hat{\boldsymbol{u}}_c) + \hat{\boldsymbol{N}}^{(2)}(\hat{\boldsymbol{u}}_c, \hat{\boldsymbol{u}}_b) \right]^* / E_a .$$

Interaction equations

The conservation laws imply some properties of the interaction coefficients. Writing

$$\mathcal{E} = \mathcal{E}^{(2)} + \mathcal{E}^{(3)} + \cdots = \frac{1}{2}\sum_{a} E_a |A_a|^2 + \frac{1}{6}\sum_{abc} S_{abc} A_a^* A_b^* A_c^* \exp(2\mathrm{i}\Omega_{abc} t) + \cdots,$$

the cubic terms in $\dot{\mathcal{E}}^{(2)} + \dot{\mathcal{E}}^{(3)} + \cdots = 0$ are equal only if

$$E_a I_a^{bc} + E_b I_b^{ca} + E_c I_c^{ab} = -2\mathrm{i} S_{abc} \Omega_{abc}.$$

Similarly, $P_a I_a^{bc} + P_b I_b^{ca} + P_c I_c^{ab} = -2\mathrm{i} T_{abc} \Omega_{abc}$.

These are useful when:

(i) $\mathcal{E}$ and $\mathcal{P}$ are exactly quadratic (cf. Rossby and internal gravity waves), or

(ii) the triad is resonant, i.e. $\omega_a + \omega_b + \omega_c = 0$.

Interaction equations

Assuming (i) or (ii),

$$\frac{E_a I_a^{bc}}{s_b - s_c} = \frac{E_b I_b^{ca}}{s_c - s_a} = \frac{E_c I_c^{ab}}{s_a - s_b} \quad \text{and} \quad \frac{P_a I_a^{bc}}{c_b - c_c} = \frac{P_b I_b^{ca}}{c_c - c_a} = \frac{P_c I_c^{ab}}{c_a - c_b},$$

where $s_a = 1/c_a = k_a/\omega_a$ is the slowness of wave a.

Assuming (ii),

$$\frac{E_a I_a^{bc}}{\omega_a} = \frac{E_b I_b^{ca}}{\omega_b} = \frac{E_c I_c^{ab}}{\omega_c} \quad \text{and} \quad \frac{P_a I_a^{bc}}{k_a} = \frac{P_b I_b^{ca}}{k_b} = \frac{P_c I_c^{ab}}{k_c}$$

This can be verified explicitly, e.g., for Rossby waves.

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Interaction equations

Remark: non-conservative systems are also governed by interaction equations of the same form, but with coefficients I_a^{bc}, I_b^{ca} and I_c^{ab} that are independent. To derive these equations, use the orthogonality

$$(2\pi)^d \boldsymbol{u}_p^+ \boldsymbol{u}_q = E_p \delta_{p,q}$$

where $\boldsymbol{u}^+$ is the left eigenvector of L.

Exercise: derive the interaction coefficients for internal gravity waves, for which

$$\boldsymbol{N}(\boldsymbol{u}) = - \begin{pmatrix} \psi_x \zeta_y - \psi_y \zeta_x \\ \psi_x b_y - \psi_y b_x \end{pmatrix},$$

with $\nabla^2 \psi = \zeta$.

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Triad interactions

The interaction equations

$$\dot{A}_a = \frac{1}{2}\sum_{bc} I_a^{bc} A_b^* A_c^* \exp(2\mathrm{i}\Omega_{abc} t)\delta_{\boldsymbol{k}_a+\boldsymbol{k}_b+\boldsymbol{k}_c},$$

are an exact reformulation of the equations of motion.
They can be approximated if we assume weak nonlinearity, i.e.,

$$A_a = O(\epsilon), \quad \epsilon \ll 1.$$

With this assumption, we can truncate the system, with the simplest truncation a wave triad obeying

$$\begin{aligned}\dot{A}_a &= I_a^{bc} A_b^* A_c^* \exp(2\mathrm{i}\Omega_{abc} t),\\ \dot{A}_b &= I_b^{ca} A_c^* A_a^* \exp(2\mathrm{i}\Omega_{abc} t),\\ \dot{A}_c &= I_c^{ca} A_a^* A_b^* \exp(2\mathrm{i}\Omega_{abc} t).\end{aligned}$$

Triad interactions

Over $t = O(\epsilon)$, the amplitudes change by $O(1)$ only in the triad is near resonant:

$$\omega_a + \omega_b + \omega_c = 2\Omega_{abc} = O(\epsilon).$$

If not, the transformation

$$A_a = B_a - \frac{\mathrm{i} I_a^{bc}}{2\Omega_{abc}} B_b^* B_c^* \exp(2\mathrm{i}\Omega_{abc} t),$$

pushes nonlinear terms to $O(\epsilon^3)$. Hence $A_a = A_a(0) + O(\epsilon)$.

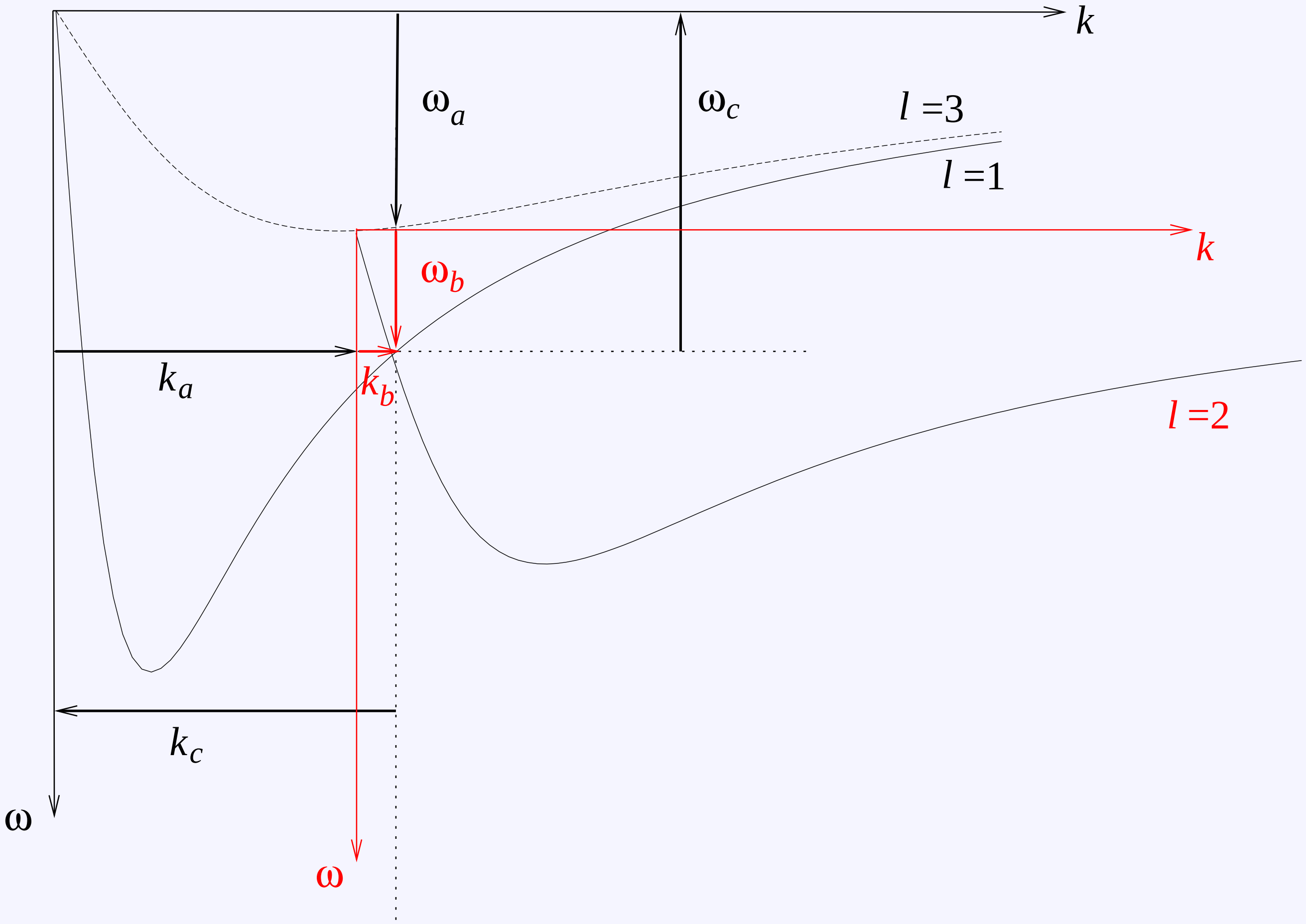
Need to consider the interaction and resonance conditions

$$vk_a + \boldsymbol{k}_b + \boldsymbol{k}_c = 0 \quad \text{and} \quad \omega_a + \omega_b + \omega_c = 0$$

$d + 1$ algebraic equations for the $3d$ unknowns

Triad interactions

Graphical solution: Rossby waves



Solution of the triad equation

Universal form: use scaled variables

$$A_a = \epsilon|I_b^{ca} I_c^{ab}|^{-1/2}\alpha_a, \quad A_b = \epsilon|I_c^{ab} I_a^{bc}|^{-1/2}\alpha_b, \quad A_c = \epsilon|I_a^{bc} I_b^{ca}|^{-1/2}\alpha_c$$

and rescale t by ϵ to find

$$\begin{aligned} \dot{\alpha}_a &= \sigma_a \alpha_b^* \alpha_c^* \exp(2\mathrm{i}\Omega t) \\ \dot{\alpha}_b &= \sigma_b \alpha_c^* \alpha_a^* \exp(2\mathrm{i}\Omega t) \\ \dot{\alpha}_c &= \sigma_c \alpha_a^* \alpha_b^* \exp(2\mathrm{i}\Omega t), \end{aligned}$$

with $\Omega = \Omega_{abc}/\epsilon = O(1)$, $\sigma_a = \mathrm{sign}\, I_a^{bc}$, etc.

Only two cases to consider:

(i) $-\sigma_a = \sigma_b = \sigma_c = 1$ or (ii) $\sigma_a = \sigma_b = \sigma_c = 1.$

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Solution of the triad equation

Manley–Rowe relations:

$$\left(\sigma_a|\alpha_a|^2 - \sigma_b|\alpha_b|^2\right)_t = \left(\sigma_b|\alpha_b|^2 - \sigma_c|\alpha_c|^2\right)_t = \left(\sigma_c|\alpha_c|^2 - \sigma_a|\alpha_a|^2\right)_t = 0$$

In case (i), amplitudes are bounded: stable;
in case (ii), amplitude can become unbounded: explosive instability.

Recall that $I_a^{bc}E_a/\omega_a = I_b^{ca}E_b/\omega_b = I_c^{ab}E_c/\omega_c$ and $\omega_a + \omega_b + \omega_c = 0$:
(ii) only possible if wave with the largest $|\omega|$ has E oppositely signed to the other two waves.
Thus, sign-definite pseudoenergy implies stable interactions (i).

Sign-indefinite pseudoenergy is possible for waves propagating on shear flows: purely nonlinear mechanism of instability.

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Solution of the triad equation

Integrate triad equations: let

$$Z(t) = \sigma_a \left[|\alpha_a(t)|^2 - |\alpha_a(0)|^2\right] = \sigma_b \left[|\alpha_b(t)|^2 - |\alpha_b(0)|^2\right]$$

$$= \sigma_c \left[|\alpha_a(t)|^2 - |\alpha_a(0)|^2\right]$$

$$W = |\alpha_a \alpha_b \alpha_c| \sin(\arg\alpha_a + \arg\alpha_b + \arg\alpha_c - 2\Omega t) + \Omega Z(t),$$

and verify $\dot{W} = 0$. Then,

$$\frac{1}{2}\dot{Z}^2 + V(Z) = 0,$$

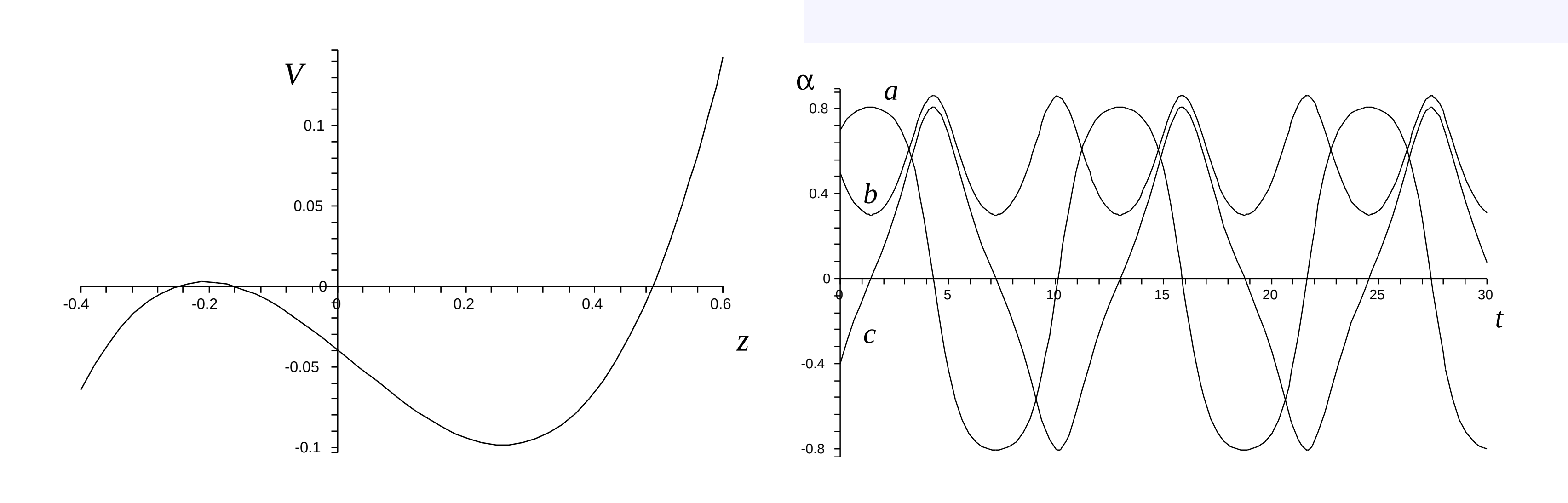
$$V(Z) = 2\left[(W - \Omega Z)^2\right.$$

$$\left.(\sigma_a Z + |\alpha_a(0)|^2)(\sigma_b Z + |\alpha_b(0)|^2)(\sigma_c Z + |\alpha_c(0)|^2)\right]$$

Dynamics of a particle with coordinate $Z(t)$, with zero energy starting at $Z(0) = 0$ in potential $V(Z)$.

Solution of the triad equation

Stable case for $\Omega = W = 0$:

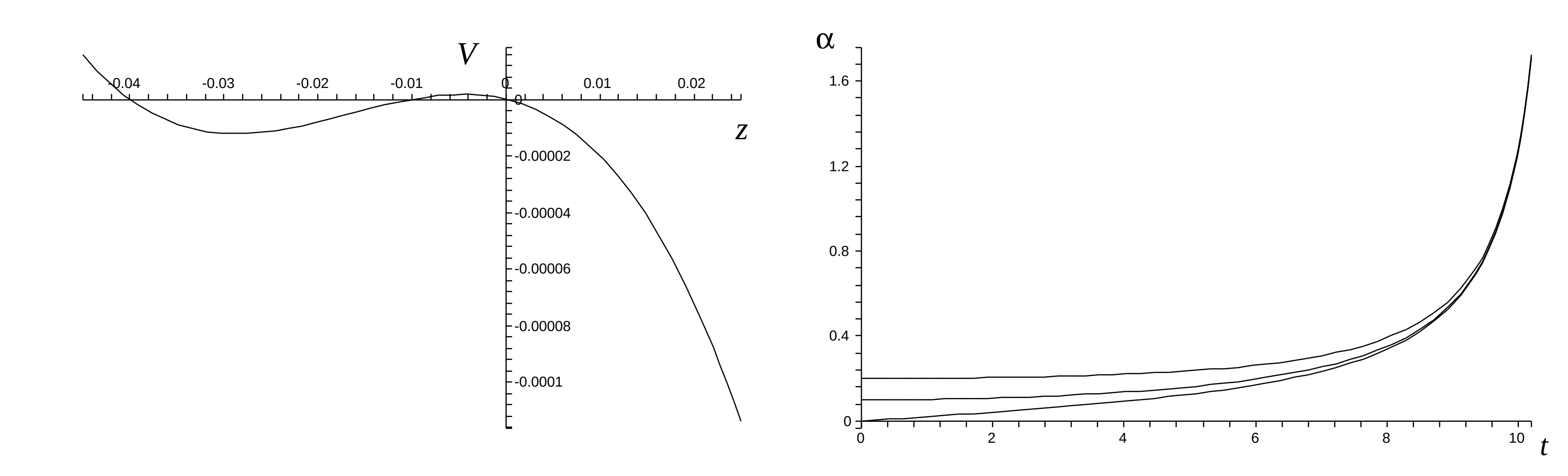


Energy exchange between the three waves, complete between a and c, partial with b.

Non-zero Ω or W inhibit energy exchanges

Solution of the triad equation

Explosive case for $\Omega = W = 0$:



Finite time blow-up of the three amplitudes: explosion

Non-zero W: slows down blow up

Non-zero Ω: slows down blow up or suppresses it for sufficiently small initial amplitudes.

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Wave instability

Consider a stable fluid, with sign-definite pseudoenergy.

Plane Rossby waves and internal gravity waves are exact nonlinear solutions of the equations of motion. Are they stable to small perturbations?

This can be studied by linear stability analysis: write

$$\zeta = A\exp[\mathrm{i}(\boldsymbol{x}\cdot vx - \omega t) + \mathrm{c.c.} + \zeta',$$

and derive a linear evolution equation for ζ'. This has periodic coefficients: need to use Floquet theory

$$\zeta' = \exp(\mu\theta)P(\theta)\exp(\mathrm{i}\boldsymbol{l}\cdot\boldsymbol{x}), \quad \theta = \boldsymbol{x}\cdot vx - \omega t,$$

for $P(\theta)$ periodic. Instability if $\Re\mu \neq 0$.

Wave instability

For small-amplitude waves, this reduces to a wave interaction problem

- wave a = primary waves whose stability is studied
- waves b and c = perturbation to wave a

Assuming $|A_b|, |A_c| \ll |A_a| \ll 1$, the interaction equations reduce to

$$\dot{A}_b = I_b^{ca} A_a^* A_c^* \exp(2\mathrm{i}\Omega_{abc} t) \quad \dot{A}_c = I_c^{ab} A_a^* A_b^* \exp(2\mathrm{i}\Omega_{abc} t)$$

where A_a is taken as a constant: pump-wave approximation.
These have solutions

$$A_b = \exp[(\lambda + \mathrm{i}\Omega)t]\hat{A}_b \quad \text{and} \quad A_c = \exp[(\lambda^* + \mathrm{i}\Omega)t]\hat{A}_c,$$

for constant $\hat{A}_b$, $\hat{A}_c$ and

$$\lambda = \pm \left[I_b^{ca} I_c^{ab} |A_a|^2 - \Omega_{abc}^2 \right]^{1/2}.$$

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Wave instability

Instability for $\Omega_{abc} = 0$, resonant triads, if

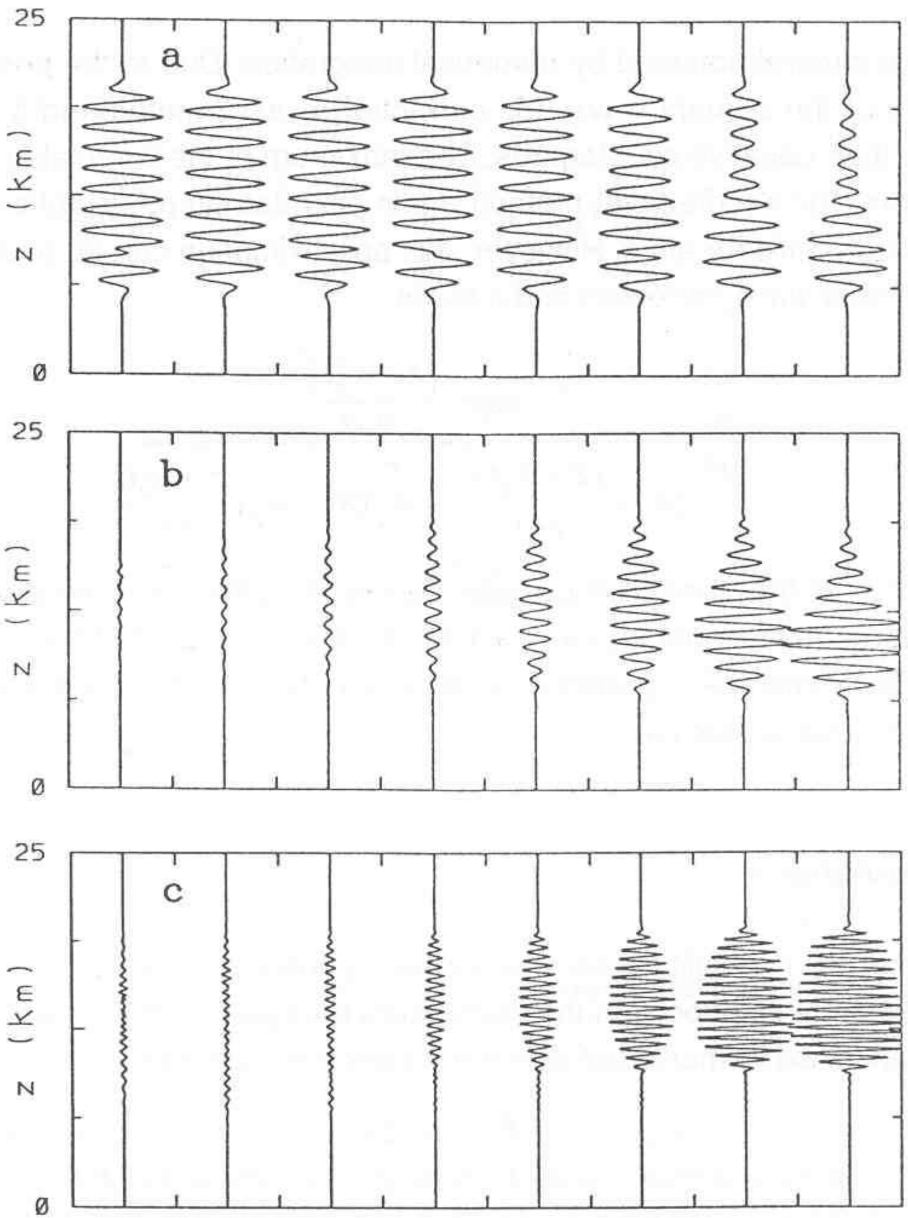
$$I_b^{ca} I_c^{ab} > 0.$$

This is the case if $|\omega_a| > |\omega_b|, \, |\omega_c|$.

Hasselmann's criterion: a wave is unstable if it is the highest frequency member of a resonant triad.
In practice, most small amplitude waves are unstable.

Example: inertia-gravity waves

Wave instability



Wave instability

Limiting cases: assume $|\boldsymbol{k}_b|, |\boldsymbol{k}_c| \gg |\boldsymbol{k}_a|$, then $\boldsymbol{k}_b \approx -\boldsymbol{k}_c$.

If b and c are on the same branch of the dispersion relation, resonance implies

$$\omega(\boldsymbol{k}_a) = -\omega(\boldsymbol{k}_b) - \omega(-\boldsymbol{k}_b - \boldsymbol{k}_a) \approx \boldsymbol{k}_a \cdot \frac{\partial \omega}{\partial \boldsymbol{k}}(\boldsymbol{k}_b).$$

In 1D, group velocity $\frac{\partial \omega}{\partial k}(k_b)$ of secondary waves matches phase velocity $\omega(k_a)/k_a$ of the primary wave.

If b and c are on branches of the dispersion relation corresponding to oppositely signed ω, resonance implies

$$\omega_b \approx \omega_c \approx -\omega_a/2.$$

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Interaction of wavepackets

Instead of plane waves, we can consider wavepackets, localized over long distances.
These are obtained by superposition of plane waves, with amplitudes that are narrowly peaked around a central wavevector, for instance

$$A(\boldsymbol{k},t) \propto \exp(-|\boldsymbol{k}-\boldsymbol{k}_a|^2/\delta^2), \quad \text{with} \;\; \delta \ll 1.$$

The wavepacket envelope is described by

$$B_a(\boldsymbol{x},t) = \int A(\boldsymbol{k}_{a'},t)\exp\left[\mathrm{i}(\boldsymbol{k}_{a'}-\boldsymbol{k}_a)\cdot\boldsymbol{x} - (\omega(\boldsymbol{k}_{a'})-\omega_a)t\right]\,\mathrm{d}\boldsymbol{k}_{a'}.$$

This is localized over a distance $|\boldsymbol{x}| = O(\delta^{-1})$.
Take $\delta = O(\epsilon)$.

Interaction of wavepackets

Consider the interaction of three wavepackets with central wavenumbers $\boldsymbol{k}_a$, $\boldsymbol{k}_b$ and $\boldsymbol{k}_c$ which are resonant.

Derive evolution equations for the corresponding envelope amplitudes $B_a(\boldsymbol{x}, t)$, $B_b(\boldsymbol{x}, t)$ and $B_c(\boldsymbol{x}, t)$.
Start with

$$\partial_t B_a(\boldsymbol{x}, t) = \int \left[\dot{A}_{a'} - \mathrm{i} A_{a'} \left(\omega_{a'} - \omega_a \right) \right] \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_{a'} - \boldsymbol{k}_a)\cdot \boldsymbol{x} - (\omega_{a'} - \omega_a)t} \, \mathrm{d}\boldsymbol{k}_{a'} .$$

Use $\omega_{a'} - \omega_a \approx \boldsymbol{c}_a \cdot (\boldsymbol{k}_{a'} - \boldsymbol{k}_a)$, with

$$\boldsymbol{c}_a = \frac{\partial \omega}{\partial \boldsymbol{k}}(\boldsymbol{k}_a),$$

and the interaction equations with $I_{a'}^{b'c'} \approx I_a^{bc}$ to find:

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Interaction of wavepackets

$$\begin{aligned}
\partial_t B_a + \boldsymbol{c}_a \cdot \nabla B_a &= I_a^{bc} B_b^* B_c^* \exp(2\mathrm{i}\Omega_{abc} t), \\
\partial_t B_b + \boldsymbol{c}_b \cdot \nabla B_b &= I_b^{ca} B_c^* B_a^* \exp(2\mathrm{i}\Omega_{abc} t), \\
\partial_t B_c + \boldsymbol{c}_c \cdot \nabla B_c &= I_c^{ca} B_a^* B_b^* \exp(2\mathrm{i}\Omega_{abc} t).
\end{aligned}$$

- Similar to amplitude equations for plane waves
- Can scale amplitudes to replace the interaction coefficients by their sign
- Partial rather than ordinary differential equations
- Additional terms: advection with the group velocity

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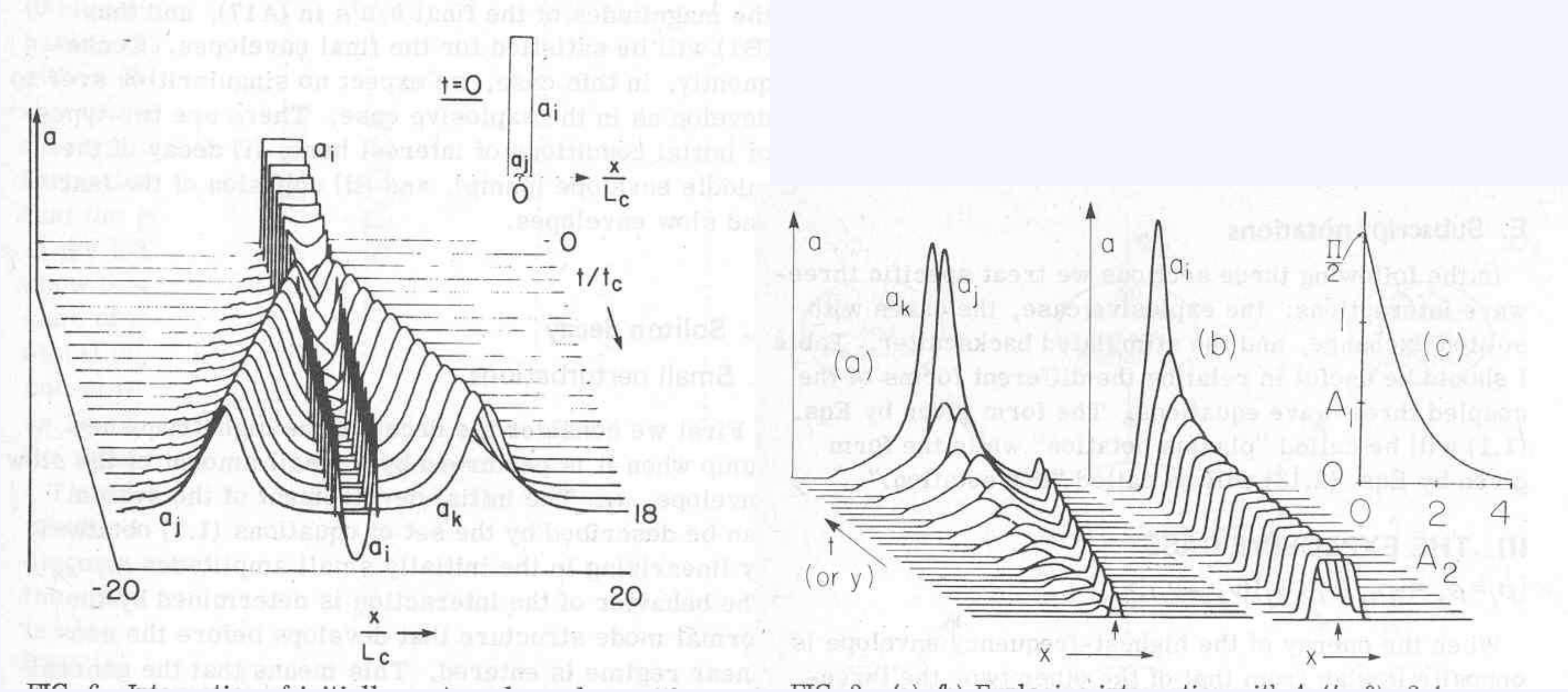
Interaction of wavepackets

The interaction equations are completely integrable by inverse scattering:

- solution consists of solitons + waves
- solitons are exchanged between envelopes a and b and c
- wave part does not disperse
- interactions are significant only if the three wavepackets collide
- energy exchanges depend on signs of interaction coefficients and relative group velocities
- explosive interaction is possible for sufficiently large initial amplitudes

Interaction of wavepackets

From Kaup et al. 1979:



stable interaction

explosive interaction

Quartet interactions

Triad interactions do not control wave interactions when

- the nonlinearity is cubic at lowest order, or
- resonant triads are impossible

In both cases, need to consider resonant quartets: four waves satisfying

$$\boldsymbol{k}_a + \boldsymbol{k}_b + \boldsymbol{k}_c + \boldsymbol{k}_d = 0 \quad \text{and} \quad \omega_a + \omega_b + \omega_c + \omega_d = 0.$$

Surface gravity waves: dispersion relation

$$\omega = \pm(g|\boldsymbol{k}|)^{1/2},$$

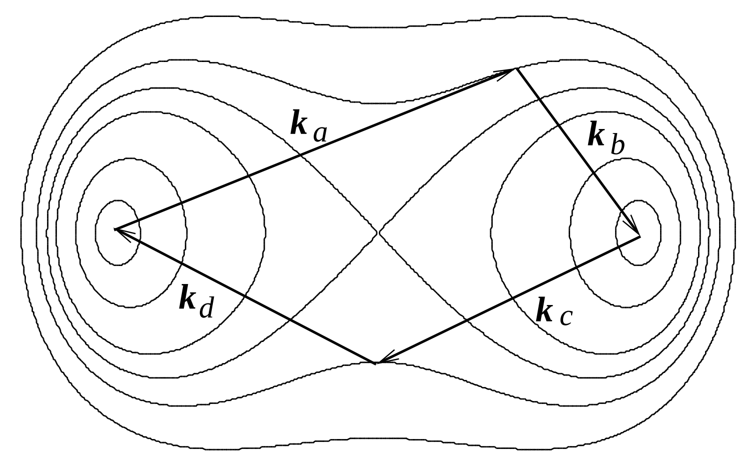
does not admit resonant triads but resonant quartets are possible.

Resonant quartets: surface waves

For surface waves, can take

$$\boldsymbol{k}_a + \boldsymbol{k}_b = -\boldsymbol{k}_c - \boldsymbol{k}_d = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then the resonance relation $|\boldsymbol{k}_a|^{1/2} + |\boldsymbol{k}_b|^{1/2} = |\boldsymbol{k}_c|^{1/2} + |\boldsymbol{k}_d|^{1/2}$ can be represented graphically in the (k_a, l_a)-plane on the level curves of $|\boldsymbol{k}_a|^{1/2} + |\boldsymbol{k}_b|^{1/2}$, with $k_b = 1 - k_a$ and $l_b = -l_c$.



Quartet interactions: interaction equations

Amplitude equations are derived as in the triad case: wave expansion and use of orthogonality. Non-resonant quadratic nonlinearities are eliminated by variable transformation. This leads to

$$\dot{A}_a = \frac{1}{6}\sum_{abcd} I_a^{bcd} A_b^* A_c^* A_d^* \exp(2\mathrm{i}\Omega_{abcd} t)\delta_{\boldsymbol{k}_a+\boldsymbol{k}_b+\boldsymbol{k}_c+\boldsymbol{k}_d}$$

A single quartet of modes (a, b, c, d) cannot be isolated: modes

$$(a, -a, a, -a), (a, -a, b, -b) \quad \text{and permutations}$$

are always resonant.
(Wave-mean flow effect: can think of $b - b \rightarrow$ mean flow followed by mean flow $- a \rightarrow a$.)

Quartet interactions: interaction equations

$$\begin{aligned}
\dot{A}_a &= \mathrm{i}A_a \sum_{j=a,d} J_a^j |A_j|^2 + I_a^{bcd} A_b^* A_c^* A_d^* \exp(2\mathrm{i}\Omega_{abcd} t), \\
\dot{A}_b &= \mathrm{i}A_b \sum_{j=a,d} J_b^j |A_j|^2 + I_b^{cda} A_c^* A_d^* A_a^* \exp(2\mathrm{i}\Omega_{abcd} t), \\
\dot{A}_c &= \mathrm{i}A_c \sum_{j=a,d} J_c^j |A_j|^2 + I_c^{dab} A_d^* A_a^* A_b^* \exp(2\mathrm{i}\Omega_{abcd} t), \\
\dot{A}_d &= \mathrm{i}A_d \sum_{j=a,d} J_d^j |A_j|^2 + I_d^{abc} A_a^* A_b^* A_c^* \exp(2\mathrm{i}\Omega_{abcd} t),
\end{aligned}$$

where $\mathrm{i}J_a^b = I_a^{(-a)b(-b)}/3$, etc.

Pseudoenergy conservation imposes that interactions coefficients purely imaginary J_a^j: for a single wave a,

$$A_a(t) = A_{a0} \exp(-\mathrm{i}J_a^a |A_{a0}|^3 t),$$

nonlinear frequency $\omega_a + J_a^a |A_{a0}|^3$.

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Quartet interactions: interaction equations

Conservation of pseudoenergy and 2 components of pseudomomentum impose

$$\frac{E_a I_a^{bcd}}{\omega_a} = \frac{E_b I_b^{cda}}{\omega_b} = \frac{E_c I_c^{dab}}{\omega_c} = \frac{E_d I_d^{abc}}{\omega_d}$$

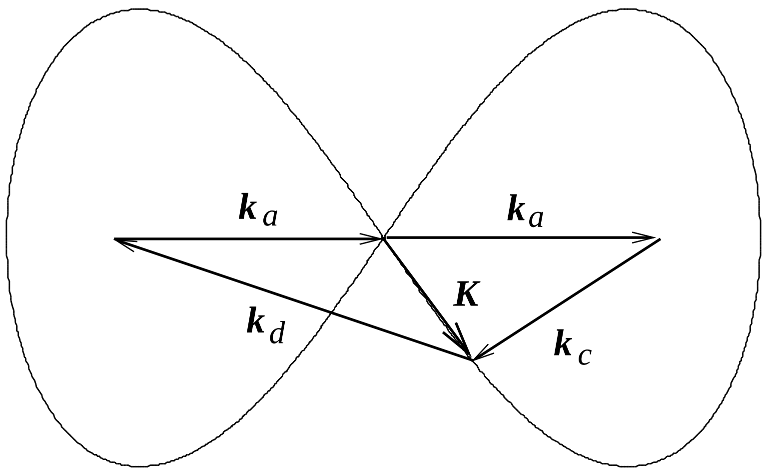
- The interaction coefficients can be reduced to signatures by scaling
- Interaction equations integrable in closed form
- Wavepackets can be considered

Quartet interactions: wave instability

Consider the stability of the primary wave a, to perturbations consisting of secondary waves c and d, with

$$\boldsymbol{k}_c = -\boldsymbol{k}_a + \boldsymbol{K} \quad \text{and} \quad \boldsymbol{k}_d = -\boldsymbol{k}_a - \boldsymbol{K}.$$

Then, a, $b = a$, c and d form a resonant quartet.



Quartet interactions: wave instability

The pump-wave approximation assumes that

$$A_a(t) = A_{a0}\exp(-\mathrm{i}J_a^a|A_{a0}|^3 t)$$

remains unchanged. The amplitudes of c and d obey the linear equations

$$\begin{aligned}\dot{A}_c &= 2\mathrm{i}J_c^a A_c A_{a0}^2 + I_c^{aad} A_d^* A_{a0}^2 \exp[2\mathrm{i}(\Omega + J_a^a A_{a0}^2)t] \\ \dot{A}_d &= 2\mathrm{i}J_d^a A_d A_{a0}^2 + I_d^{aac} A_c^* A_{a0}^2 \exp[2\mathrm{i}(\Omega + J_a^a A_{a0}^2)t].\end{aligned}$$

Solutions can be sought in the form

$$\begin{aligned}A_c &= \hat{A}_c \exp[\mathrm{i}(\Omega + J_a^a A_{a0}^2)t]\exp(\lambda t) \\ A_d &= \hat{A}_d \exp[\mathrm{i}(\Omega + J_a^a A_{a0}^2)t]\exp(\lambda^* t)\end{aligned}$$

for constant $\hat{A}_c$, $\hat{A}_d$ and growth rate λ.

Quartet interactions: wave instability

Growth rate

$$\lambda = \mathrm{i}(J_a^c - J_a^b) \pm \left[I_c^{aad} I_d^{aac} A_{a0}^4 - [\Omega + (J_a^a - J_c^a - J_d^a)A_{a0}^2]^2\right]^{1/2}.$$

Instability is possible provided that $\Omega/A_{a0}^2 + (J_a^a - J_c^a - J_d^a)$ is small enough.

Side-band instability: limiting case where $|\boldsymbol{K}| \ll |\boldsymbol{k}_a|$, i.e. the perturbation represents a large-scale modulation of the amplitude of a (cf. NLS equation).

In this limit,

$$\lambda = \left[2\Omega I A_{a0}^2 - \Omega^2\right]^{1/2} \quad \text{with} \quad \Omega = -\frac{1}{2}\boldsymbol{K} \cdot \frac{\partial^2 \omega}{\partial \boldsymbol{k} \partial \boldsymbol{k}}(\boldsymbol{k}_a) \cdot \boldsymbol{K}.$$

Instability occurs if $\Omega(I A_{a0}^2 - \Omega/2) > 0$: condition in terms of nonlinear frequency shift I and linear dispersion relation.

Alternative derivations: Lagrangian systems

Use the Lagrangian or Hamiltonian structure to derive interaction equations, with explicitly symmetric interaction coefficients.

Lagrangian system: derived from variational principle

$$\delta \int \mathrm{d}t \int \mathrm{d}\boldsymbol{x}\, L(\boldsymbol{u}, \dot{\boldsymbol{u}}, \nabla \boldsymbol{u}, \cdots) = 0,$$

Example: Klein–Gordon equation $\psi_{tt} - \psi_{xx} + \psi + 4\sigma\psi^3 = 0$ follows from

$$\delta \int \left[\psi_t^2 - \psi_x^2 - \psi^2 - 2\sigma\psi^4\right] \mathrm{d}x = 0.$$

To derive interaction equations, introduce

$$\psi = \sum_a A_a(t) \exp[\mathrm{i}(\boldsymbol{k}_a \cdot \boldsymbol{x} - \omega_a)t].$$

Alternative derivations: Lagrangian systems

The variational principle becomes

$$\delta\left(\mathcal{L}^{(2)} + \mathcal{L}^{(3)} + \cdots\right) = 0,$$

where the variations are taken with respect to the amplitudes A_a.

Find

$$\mathcal{L}^{(2)} = \sum_a D(\boldsymbol{k}_a, \omega_a)|A_a|^2,$$

The leading-order variation gives

$$D(\boldsymbol{k}_a, \omega_a) = 0,$$

a form of the dispersion relation.

At next order (with $\dot{A}_a = O(\epsilon)$), integrate over t keeping $\Omega_{abc}t$ fixed:

$$\mathcal{L}^{(3)} = \mathrm{i}\int \mathrm{d}t \left[\frac{1}{2}\sum_a D_\omega(\boldsymbol{k}_a, \omega_a)\dot{A}_a A_a^* - \frac{1}{6}\sum_{abc} \Gamma^*_{abc} A_a A_b A_c \exp(-2\mathrm{i}\Omega_{abc}t)\right]$$

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Alternative derivations: Lagrangian systems

Taking variations give

$$D_\omega(\boldsymbol{k}_a, \omega_a)\dot{A}_a = \frac{1}{2}\sum_{bc} \Gamma_{abc} A_b^* A_c^* \exp(2\mathrm{i}\Omega_{abc} t)\delta_{\boldsymbol{k}_a+\boldsymbol{k}_b+\boldsymbol{k}_c}.$$

Interaction equations, with

$$I_a^{bc} = \frac{\Gamma_{abc}}{D_\omega(\boldsymbol{k}_a, \omega_a)}.$$

Since mode pseudoenergy is $E_a = \omega_a D_\omega(\boldsymbol{k}_a, \omega_a)$, the interaction coefficients explicitly satisfy

$$\frac{E_a I_a^{bc}}{\omega_a} = \frac{E_b I_b^{ca}}{\omega_b} = \frac{E_c I_c^{ab}}{\omega_c}.$$

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Hamiltonian formulation

For many fluid systems, the Lagrangian formulation is not the most natural one: in usual coordinates, the systems are non-canonical Hamiltonian systems.

Rossby waves: $\zeta_t + \beta\psi_x + \partial(\psi, \zeta)$ can be written

$$\zeta_t = J\frac{\delta\mathcal{E}}{\partial\zeta}, \quad J = -\partial(\zeta, \cdot).$$

The derivation of interaction equations is simple only for canonical systems: J is constant.

Zakharov–Piterbarg: introduce a near-identity transformation of ζ to make the system canonical. Let $\tilde{\zeta}$ be

$$\tilde{\zeta} = \zeta + \zeta\zeta_y/\beta + O(\zeta^3).$$

Hamiltonian formulation

The equation of motion becomes

$$\tilde{\zeta}_t + \beta\partial_x(\psi + \psi_y\tilde{\zeta}) = O(\epsilon^3).$$

This has the canonical Hamiltonian form

$$\tilde{\zeta}_t = \beta\partial_x \frac{\partial \mathcal{H}}{\delta\tilde{\zeta}}, \quad \text{with} \quad \mathcal{H} = \frac{1}{2}\int |\nabla\psi|^2 \, \mathrm{d}\boldsymbol{x} = -\frac{1}{2}\int \psi\zeta \, \mathrm{d}\boldsymbol{x}.$$

Introducing the expansion

$$\tilde{\zeta} = \sum_a C_a(t) \exp(\mathrm{i}\boldsymbol{k}_a \cdot \boldsymbol{x}) \quad \text{with} \quad C_a = \frac{1}{(2\pi)^2}\int \tilde{\zeta}(\boldsymbol{x}, t) \exp(-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x},$$

the equations of motion follow from

$$\dot{C}_a = -\frac{\mathrm{i}\beta k_a}{(2\pi)^2}\frac{\partial \mathcal{H}}{\partial C_a^*}.$$

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Hamiltonian formulation

From

$$
\begin{aligned}
\zeta &= \sum_a C_a(t)\exp(\mathrm{i}\boldsymbol{k}_a\cdot\boldsymbol{x}) - \frac{\mathrm{i}}{2\beta}\sum_{bc}(l_b+l_c)C_bC_c\exp[\mathrm{i}(\boldsymbol{k}_b+\boldsymbol{k}_c)\cdot\boldsymbol{x}],\\
\psi &= \sum_a \frac{-1}{k_a^2+l_a^2}C_a(t)\exp(\mathrm{i}\boldsymbol{k}_a\cdot\boldsymbol{x})\\
&+ \frac{\mathrm{i}}{2\beta}\sum_{bc}\frac{l_b+l_c}{(k_b+k_c)^2+(l_b+l_c)^2}C_bC_c\exp[\mathrm{i}(\boldsymbol{k}_b+\boldsymbol{k}_c)\cdot\boldsymbol{x}],
\end{aligned}
$$

the Hamiltonian follows as

$$
\mathcal{H} = \frac{(2\pi)^2}{2}\left(\sum_a \frac{|C_a|^2}{k_a^2+l_a^2} + \frac{\mathrm{i}\beta}{3}\sum_{abc}C_a^*C_b^*C_c^*\right),
$$

$$
\Delta_{abc} = -\left(\frac{l_a}{k_a^2+l_a^2}+\frac{l_b}{k_b^2+l_b^2}+\frac{l_c}{k_c^2+l_c^2}\right).
$$

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Hamiltonian formulation

The interaction equations are then

$$\dot{C}_a = \mathrm{i}\omega_a C_a + k_a \sum_{bc} \Delta_{abc} C_b^* C_c^* .$$

With $C_a = A_a \exp(-\mathrm{i}\omega_a t)$, these are equivalent to those found in Lecture 1 near resonance since

$$k_a \Delta_{abc} \approx \frac{l_a k_b}{k_b^2 + l_b^2} + \frac{l_a k_c}{k_c^2 + l_c^2} - \frac{k_a l_b}{k_b^2 + l_b^2} - \frac{k_a l_c}{k_c^2 + l_b^2} = I_a^{bc}$$

where I_a^{bc} is given in .

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Waves in shear flow

When equations of motion depend on a coordinate, y, waves take the modal form

$$\boldsymbol{u} = A\hat{\boldsymbol{u}}(y)\exp[\mathrm{i}(kx - \omega t).$$

The dispersion and polarisation relations are found by solving a differential eigenvalue problem.

Example: Rossby waves in a shear flow $(U(y), 0)$ obey the linearized equation

$$(\partial_t + U\partial_x)\zeta + (\beta - U'')\partial_x\psi = 0, \quad \nabla^2\psi = \zeta,$$

conserving

$$\mathcal{E}^{(2)} = \frac{1}{2}\int \left[|\nabla\psi|^2 - U\zeta^2/(\beta - U'')\right]\,\mathrm{d}\boldsymbol{x}.$$

and

$$\mathcal{P}^{(2)} = -\frac{1}{2}\int \left[\zeta^2/(\beta - U'')\right]\,\mathrm{d}\boldsymbol{x}.$$

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Waves in shear flow

The eigenvalue problem

$$(U-c)(\hat{\psi}''-k^2\hat{\psi})+(\beta-U'')\hat{\psi}=0$$

with $\hat{\psi}=0$ for $y=0,L$ has two types of solutions:

- Rossby waves for $c=c_n,\ n=1,2,\cdots$,
- singular modes for $U_{\text{min}}<c<U_{\text{max}}$, singular at $y_c\ :\ U(y_c)=c$, where

$$\left[\frac{\mathrm{d}\psi}{\mathrm{d}y}\right]_{y_c^-}^{y_c^+}=\lambda_c,\quad \zeta\sim\lambda_c\delta(y-y_c)+\cdots$$

Superposition of singular modes is smooth and represents a sheared disturbance:

$$\zeta=\int\hat{A}(t;c)\zeta(y;c)\mathrm{e}^{\mathrm{i}k(x-ct)}\,\mathrm{d}c\sim A(t;U(y))\mathrm{e}^{\mathrm{i}k(x-U(y)t)}.$$

Waves in shear flows

Orthogonality relations remain valid, although in a generalized sense for singular modes.
For fixed $\boldsymbol{k}$, pseudomomentum orthogonality,

$$\int \frac{\hat{\zeta}_a(y)\hat{\zeta}_b(y)}{\beta - U''(y)} \mathrm{d}y = P_a \delta_{a,b}$$

for regular modes $a, b = 1, 2, \cdots$, and

$$\int \frac{\hat{\zeta}(y; c_a)\hat{\zeta}(y; c_b)}{\beta - U''(y)} \mathrm{d}y = P(c_a)\delta(c_a - c_b)$$

for singular modes with $U_{\text{min}} < c_a, c_b < U_{\text{max}}$.

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Waves in shear flow

Interaction equations can be derived formally but

- presence of terms secular in t ($\partial_y \zeta \sim t$ for sheared disturbances
- can be remedied by expanding $\tilde{\zeta} = \zeta - \zeta \partial_y \zeta / (\beta - U'')$
- cannot truncate the continuous spectrum of singular modes
- simple truncations break down for $t = O(\epsilon^{-1})$

Open problem: explosive interaction in shear flows (must involve singular modes).